Application of Coherence Function to the Analysis of Compressive Sensing

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***Abstract*—Compressive sensing has been developed for the sampling of sparse or compressible signals. Strong theorems state that when a signal is sufficiently sparse, its samples can be accurately recovered from random sub-Nyquist measurements. As a consequence, compressive sensing is emerging as a part of various applications, such as image processing, biomedical problems or audio signal processing. Designing a compressive sensing application comprises the selection of many parameters, e.g. data acquisition scheme, compression ratio, reconstruction algorithm, etc. To make these decisions experimentally, a simple criterion to compare several options can prove to be helpful. This paper proposes to use the coherence function as a criterion to evaluate the quality of a signal transmission via compressive sensing. After a brief review of compressive sensing, the usage of the coherence function is presented. Simulation examples illustrate how it can help making the design decisions.**

***Index Terms*—coherence function, compressive sensing, FFT, stochastic signals**

I. Introduction

Traditionally, sampling is governed by Shannon's theorem. This well-known result is universal, it can be used for sampling any signal. In practice, many signals can be described with only a few significant coefficients in an appropriate basis, frame or dictionary (for brevity, in the following only the word basis will be used). This phenomenon is called sparsity.

A signal is sparse if there is a basis in which it has few nonzero coefficients. Similarly, a signal is compressible in a basis if its sorted coefficients decay rapidly (enveloped by an exponential decay). Whether a signal is sparse (compressible) or not, depends on the basis. To illustrate this, one can consider the (inverse) discrete Fourier transform of a single spike. A basis in which a signal has a sparse representation, is called the sparsifying basis (for that signal).

Compressive sensing was introduced in 2004 by Donoho, Candès, Romberg and Tao [1], [2], [3] for the sampling of sparse or compressible signals. Traditionally, using Shannon's theorem, one would take a number of samples, and then use a compression algorithm to represent the signal with a fewer number of samples. Compared to the sparsity of the signal, one oversamples it, then performs the compression and only keeps the significant coefficients. Thus, a great part of the acquired data is discarded. In contrast, using compressive sensing, one directly obtains a compressed representation via random sampling. Sampling and compression are performed simultaneously, at a sub-Nyquist rate.

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One would expect that if the sampling is sub-Nyquist, the signal cannot be reconstructed exactly. For general signals, this is true. However, for sparse signals compressive sensing offers accurate reconstruction from sub-Nyquist measurements using nonlinear reconstruction algorithms [4], [5], [6].

Applications of compressive sensing are emerging in various fields of science and technology. A famous image processing example is the single-pixel camera [7]. Some other fields are biomedical problems [8], or face recognition [9]. For a broad overview of compressive sensing acquisition and reconstruction strategies, as well as applications we refer the readers to the survey paper [10].

When one designs an application of compressive sensing, there are multiple decisions to make. A major task is to determine the sparsifying basis. Moreover, one needs to decide the data acquisition and the reconstruction schemes. They can have several parameters to tune, the most trivial is the rate of compression. These decisions may require extensive knowledge about the compressive sensing structures and algorithms.

Another way of making these design decisions is via experimentation. In many applications, the signals to be processed can be modeled well by stochastic signals, e.g. noise or vibration signals; nonstationary signals; audio, acoustic and speech signals or signals containing short periodic parts. Furthermore, in many applications, the signals' frequency domain behavior is technically relevant. In these cases, it is important to accurately transmit those frequency bands which contain the signal. When such stochastic signals are transmitted through a system, the transmission quality can be assessed in the frequency domain by calculating the coherence function between the input and output signals. We propose to use the coherence function in order to help making the design decisions by experimentation.

The paper is arranged as follows: Section II gives an overview of compressive sensing. The usage of the coherence function is discussed in Section III. Section IV presents some simulation examples. The paper concludes in Section V.

II. Compressive Sensing

Compressive sensing can be split into two tasks:

- Data acquisition: getting the compressed measurements from the input signal.
- Reconstruction: getting the estimate of the input signal from the compressed measurements.

In the following, these tasks are reviewed briefly.

A. Data Acquisition

Data acquisition can be modeled as follows:

$$y = \varphi x \tag{1}$$

where $x \in \mathbb{C}^n$ is the input vector, $\varphi \in \mathbb{C}^{m \times n}$ is the measurement matrix and $y \in \mathbb{C}^m$ is the vector of compressive measurements.

Usually φ is chosen as a random matrix, e.g. with elements drawn from a Gaussian distribution. Furthermore, $m = \tau n$, where $0 < \tau \leq 1$ is the compression ratio [11]. $\tau < 1$ means $m < n$. In certain applications, even $m \ll n$ can be achieved. The actual value of τ depends on the sparsity of the signal. To obtain a low value, the measurement matrix needs to be incoherent with the sparsifying basis [12].

B. Reconstruction

In the reconstruction problem, we are given the measurements y, the measurement matrix φ, and we try to solve the measurement equation (1) for x. This is an underdetermined system with infinitely many solutions. The usual least squares approach yields poor results, since it tries to give a solution with minimal energy, disregarding the sparsity of the signal.

If $\psi \in \mathbb{R}^{n \times k}$ is the sparsifying basis of the signal, that is

$$x = \psi s \tag{2}$$

where $s \in \mathbb{R}^k$ (or $\mathbb{C}^k$), $k \geq n$ is the sparse (or compressible) coefficient vector, then the measurement equation (1) can be rewritten as

$$y = \varphi \psi s = \Theta s. \tag{3}$$

Since we know that s is a sparse vector, the unique solution of the reconstruction problem can be determined by choosing the sparsest possible s. Mathematically, this is described by the following l_0 optimization:

$$\hat{s} = \arg\min_s \|s\|_0 \quad \text{subject to} \quad y = \Theta s \tag{4}$$

where $\hat{s}$ is the estimated coefficient vector and $\|\cdot\|_0$ denotes the l_0 pseudonorm which is the number of nonzero elements. Now the input signal can be estimated:

$$\hat{x} = \psi \hat{s}. \tag{5}$$

Directly solving (4) is computationally extensive, since it involves trying all the possible combinations, which is an NP-hard problem. Several alternate methods have been proposed in the literature, e.g. to use convex optimization (l_1 norm):

$$\hat{s} = \arg\min_s \|s\|_1 \quad \text{subject to} \quad y = \Theta s \tag{6}$$

This modified reconstruction problem can be solved using linear programming techniques. This solution is called Basis Pursuit [4].

III. Coherence Function

The $\gamma^2(f)$ coherence function [13] between signals x and z is defined as

$$\gamma_{xz}^2(f) = |S_{xz}(f)|^2 / (S_{xx}(f) S_{zz}(f)) \tag{7}$$

where $S_{xx}(f)$ and $S_{zz}(f)$ are the auto power spectral densities, while $S_{xz}(f)$ is the cross power spectral density. In practice, they can be efficiently estimated using FFT. Note that this coherence function is a different concept than the (in)coherence between the φ measurement and the ψ sparsifying matrices [14] used in the field of compressive sensing.

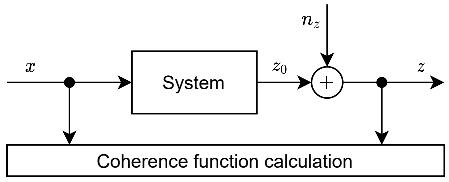

Fig. 1. Setup used for the coherence function example

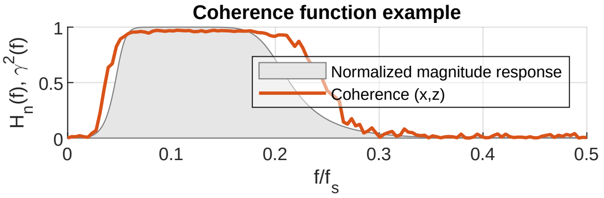

Fig. 2. Coherence function example. Grey area: normalized magnitude response, $|H|_{\text{max}} = 1$. Red line: coherence between signals x and z.

At each frequency, the coherence function indicates the correlation between signals x and z. That is, $0 \leq \gamma_{xz}^2(f) \leq 1$ and a high value indicates a linear relationship between signals x and z. The coherence is decreased in the presence of uncorrelated noise or nonlinearities in the system.

The following demonstrative example illustrates the application of the coherence function for assessing the quality of a signal transmission.

Fig. 1 depicts the setup: first, 10000 samples of Gaussian white noise are generated (signal x). Then, it is processed by a 4th order Butterworth bandpass filter (passband: $[0.05 \dots 0.2] \cdot f_s$, where f_s is the sampling frequency) to obtain z_0. Then the n_z Gaussian white noise is added to the output with SNR $= 10$ dB to get the z output signal. Finally, the coherence function is calculated for the input against the noisy output.

The coherence function is shown in Fig. 2 alongside the $H_n(f) = |H(f)| / |H|_{\text{max}}$ normalized magnitude response of the system ($|H|_{\text{max}}$ is the maximal magnitude response). The coherence between the noisy output and the input is drawn with red color, while the gray area is the magnitude response of the filter. To make the coherence function easily comparable to the magnitude response, the magnitude response is drawn as an area, on a linear scale.

The coherence between the noisy output and the input signal is as one would expect: it is high (low) when the magnitude response is high (low). Since the spectrum at the input x is white, the spectrum of z_0 is shaped like the magnitude response. Because of the noise n_z, the output z is dominated by the noise in the stopband, where the magnitude response is low. As this noise is uncorrelated with the input, it is not contained in the $S_{xz}(f)$ cross spectrum, but contained in the $S_{zz}(f)$ output spectrum. Thus, the numerator of (7) is unchanged, while its denominator grows: the coherence is decreased. Similar arguments can be made for the passband.

A potential application of compressive sensing is compressing the transmitted data. In many cases, the signals to be transmitted can be modeled well by stochastic signals. Some examples are noise measurement [15], vibration analysis [16],

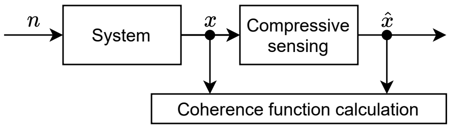

Fig. 3. Setup used for the simulations

or signals containing short periodic parts such as speech [17], acoustic or in general audio signals.

In these cases, the transmission utilizing compressive sensing can be modeled as the processing of a stochastic signal with a system. For this model, a possible way of assessing the quality of the transmission is to calculate the coherence function between its input and output signals. We propose to use the coherence function in order to help making the design decisions by experimentation.

A. Alternative Metrics

There are several metrics to evaluate a compressive sensing scheme [11]. E.g. two popular ones are the Normalized Root Mean Square Error (NRMSE) and the Signal-to-Error Ratio (SER):

$$\text{NRMSE} = \|x - \hat{x}\|_2 \,/\, \|x\|_2 \tag{8}$$

$$\text{SER} = -20 \lg \text{NRMSE} \tag{9}$$

Note that while the NRMSE is a normalized metric, it can obtain values higher than one (since the normalization refers to the division by $\|x\|_2$).

The usual metrics give a scalar, integral measure of the quality from time domain analysis. In contrast, the coherence function provides a vector of quality metrics in the frequency domain.

IV. Examples

To illustrate how the coherence function can help designing a signal transmission using compressive sensing, some simulation examples are presented.

Thus the aim of the examples below is not to illustrate the power of an optimized data transmission using compressive sensing, but to present how the coherence function shows the difference between various compressive sensing schemes. As a consequence, not necessarily the best sparsifying bases or the most powerful reconstruction algorithms are used.

A. The Simulation Environment

The setup used for the simulations is shown in Fig. 3. To generate the x input signal, the n Gaussian white noise is processed with a system. Then this x signal is passed through the compressive sensing data acquisition–reconstruction scheme to obtain the $\hat{x}$ estimate of the input signal.

The elements of the φ measurement matrix are drawn from a Gaussian distribution. The τ compression ratio was varied from 5% to 50% in steps of 5%. The l_1-magic implementation [18] of solving (6) is used for the reconstruction. For simplicity, only changes in τ and in the sparsifying basis are shown in the examples. However, the proposed analysis is applicable e.g. to changes in the reconstruction algorithm as well.

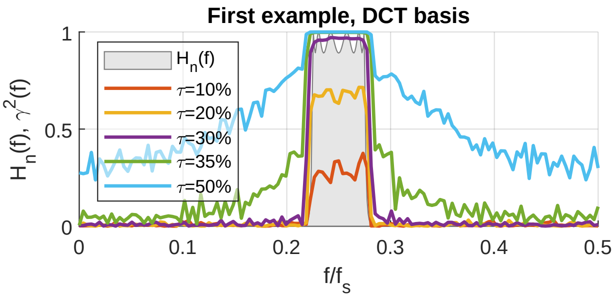

Fig. 4. First example, DCT basis. Grey area: normalized magnitude response, $|H|_{\max} = 1$. Colored lines: coherence function at different compression ratios.

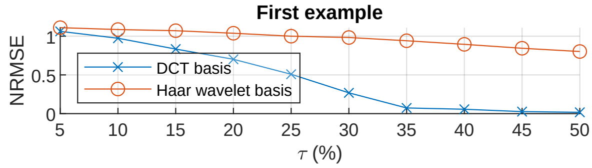

Fig. 5. NRMSE with different compression ratios in the first example

B. Example 1: Elliptic Bandpass Filter

In the first example, the system is an elliptic bandpass filter (6th order, 1 dB passband ripple, 60 dB stopband attenuation, $[0.45 \ldots 0.55] \cdot f_s/2$ passband). Its magnitude response is shown in Fig. 4 as the grey area. Consequently, the signal x should contain significant coefficients only in the passband, that is, x should be approximately sparse.

The discrete cosine transform (DCT) [19] is selected for the sparsifying basis as an initial choice. The results are shown in Fig. 4 with the colored lines. As the compression ratio increases, the coherence increases first in the passband, then also in the stopband. The coherence function takes almost 1 values in the passband at $\tau \geq 35\%$. From this, one could infer that $\tau = 35\%$ is enough for a good transmission.

This claim can be verified by looking at the NRMSE. Fig. 5 shows this error for the investigated τ values. The error decreases rapidly from 30% to 35%, while from 35% to 50% it is still decreasing, but at a slower rate. There is a clear break in the NRMSE where the coherence function fully "envelopes" the input spectrum.

From the perspective of the designer, plotting the input spectrum and the coherence function is more informative than calculating a single number.

Other potential sparsifying bases can be tried and evaluated, here the results of the Haar wavelet basis [20] are presented (Fig. 6). Comparing to Fig. 4, it is clear that the coherence at a given τ is worse for the Haar wavelet basis than for the DCT basis. This also can be seen on the NRMSE plot. Consequently, the DCT basis is a better choice than the Haar wavelet basis in this example.

Note that for both bases, the coherence function has higher values in the passband and lower values in the stopband. This is as expected: as τ grows, less and less significant parts of the signal are getting transmitted also. When the coherence is high in all the significant bands, the useful information in the signal are transmitted. This is harder to see in the time domain error plot.

A potential remark would be the idea to transform this

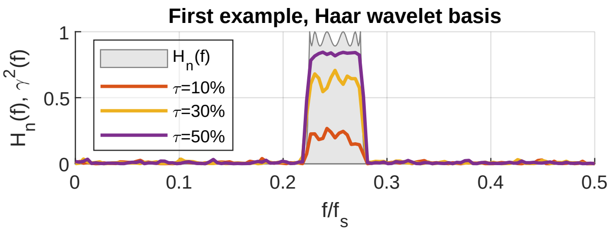

Fig. 6. First example, Haar wavelet basis. Gray area: normalized magnitude response, $|H|_{\max} = 1$. Colored lines: coherence function at different compression ratios.

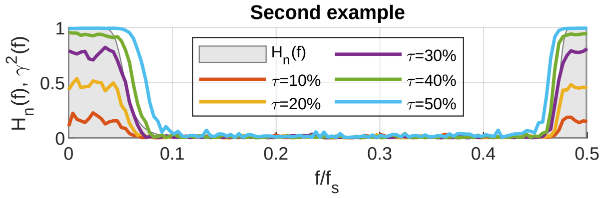

Fig. 7. Second example, DCT basis. Grey area: normalized magnitude response, $|H|_{\max} = 1$. Colored lines: coherence function at different compression ratios.

signal to baseband and there use traditional sampling. In this simple case, this is a viable solution. However consider a case when the signal has several bands scattered in the spectrum, with the same "total bandwidth" as here. In such a case, compressive sensing requires similar compression ratio as here, while generally the transformation to baseband is not applicable.

C. Example 2: Butterworth Bandstop Filter

The system in the second example is a Butterworth bandstop filter (6th order, $[0.1 \dots 0.95] \cdot f_s/2$ stopband). Fig. 7. illustrates its normalized magnitude response with the gray area. In the first example, there was a single, narrow passband. Here, the passband is still narrow, but is split into two bands. Similarly to the first example, x should be approximately sparse.

After calculating the coherence function with the DCT basis, we got the results shown in Fig. 7. Compared to the previous example, at first glance now we can see that $\tau = 50\%$ is required to reach the coherence value of 1 in the passbands. This is larger than there, however, the total width of the passbands is also larger than in the first example.

The time domain analysis shows that $\tau = 50\%$ transmission offers similar quality to the $\tau = 35\%$ case in the first example (Fig. 8). Again, the coherence is higher in the passbands and lower in the stopband.

V. Conclusion

In this paper the usage of coherence function was proposed to assess the transmission quality of stochastic signals via compressive sensing. After taking an overview of the compressive sensing process, the coherence function was reviewed. In many signal processing applications, the signals' frequency domain behavior is technically relevant. Thus, it is important to accurately transmit those frequency bands which contain the majority of the signal's power. When such stochastic signals are transmitted through a system, the coherence function can be used as a tool to compare the quality of different data transmission options. Simulation examples illustrated the usage of coherence function to qualify a signal transmission via compressive sensing. The results showed that in certain simple cases, similar compression can be reached with compressive sensing as with traditional methods. A potential future task is finding such examples which can better illustrate the usage of the coherence function.

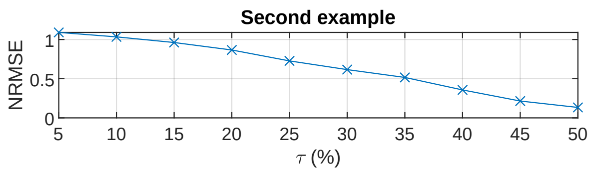

Fig. 8. NRMSE with different compression ratios in the second example

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