21th IMEKO TC4 International Symposium on Understanding
the World through Electrical and Electronic Measurement,
and 19th International Workshop on ADC Modelling and Testing
Budapest, Hungary, September 7-9, 2016

Parameterization of Nonlinearity for Efficient Estimation in ADC Testing

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Abstract – **In Maximum Likelihood (ML) estimation of ADC parameters, the estimation of static transfer characteristic has key importance. However, the description of static transfer characteristic demands to handle numerous values. Without parameterization, the nonlinearity of an N-bit converter can be described using $2^N - 1$ integral nonlinearity (INL) values. Nevertheless for real ADCs the INL values are not independent, the information regarding the nonlinearity can be compressed. This paper enumerates multiple methods to measure and approximate the static transfer characteristic of the ADCs, and evaluates their efficiency. The results are expressed paying attention to the number of parameters and using standard measures for the approximation error (e.g. the l_∞ norm of the error vector).**

Keywords— ADC testing, parameterization, INL, estimation, maximum likelihood

I. INTRODUCTION

In ADC testing with sinusoidal excitation, the most important task is to estimate the parameters of the analog signal (which is assumed to be a sine wave and additional noise). Using the parameter estimators of the excitation, the static and dynamic quality parameters (such as INL, DNL, SINAD, ENOB, etc.) of the ADC under test can be calculated via simple, closed-form calculations. The most robust and straightforward way to estimate the signal parameters is to perform a sine wave fit in least squares (LS) sense (standards IEEE-1241 [1] and IEEE-1057 [2] recommend to do so). However, these LS estimators can be biased, do not provide minimal variance, and even LS fitting algorithm can fail to converge properly (strongly depending on the initial frequency estimation). The main disadvantage of LS estimation is bias: if the nonlinearity of the converter is significant compared to the amount of noise on the analog signal, the LS estimators do not reach the real value of the parameter, but vary near an other one. The idea of using maximum likelihood estimation for ADC testing provides a solution for this problem: as ML estimators are consistent, the signal parameters can be estimated with arbitrary low bias increasing the number of observations (in this case the number of samples in the measurement record) [3]. Nevertheless the practical realization of ML estimations propounds several problems, and the hardest challenge is the exponential growth of the parameter space depending on the number of bits. The general likelihood function of the observations [4] uses $2^b + 4$ parameters, where b denotes the number of bits. Four parameters describe the analog sine wave (A, B, C and f for cosine coefficient, sine coefficient, DC component and frequency respectively), one parameter denotes the standard deviation of the additive noise on the analog signal (σ) and $2^b - 1$ parameters ($T[1]$, $T[2]$, ...,$T[2^b - 1]$) describe the code transition levels of the device under test. This way the ML estimators for the signal and ADC parameters can maximize the likelihood function with respect to $2^b + 4$ parameters:

$$L(A, B, C, f, \sigma, T[1], T[2], ..., T[2^b-1]) = \prod_{k=1}^{M} P[y_k = Y_k] \tag{1}$$

where y_k denotes the k^{th} sample of the measurement record and Y_k is the discrete random variable that represents the probability distribution of the k^{th} sample, assuming the given signal, noise and ADC parameters. Thus

$$\hat{\mathbf{p}}_{\mathrm{ML}} = \underset{\mathbf{p}}{\operatorname{argmax}}\, L(\mathbf{p}) \tag{2}$$

where $\mathbf{p}$ denotes the full parameter vector:

$$\mathbf{p}^{\mathrm{T}} = \begin{bmatrix} A & B & C & f & \sigma & T[1] & T[2] & \dots & T[2^b - 1] \end{bmatrix} \tag{3}$$

maximizing an objective function with respect to e.g. 65539 parameters (in case of very common, 16-bit converter) propounds practically unsolvable numerical problems and requires computation efforts that cannot be realized in PC environment (or the time of estimation increases to unacceptable values). Reducing the parameter space is essential to provide - approximate - maximum likelihood signal and ADC parameter estimators with passable efforts.

II. CURRENT SOLUTIONS

The most straightforward way to decrease the number of parameters in the objective function is to estimate the transition levels from the sinusoidal record via histogram test [5]. These transition level estimators will be handled as fix values and shall not be modified during the optimization. This way the likelihood function is optimized with respect to five parameters.

$$[\hat{A}_{\mathrm{ML}}, \hat{B}_{\mathrm{ML}}, \hat{C}_{\mathrm{ML}}, \hat{f}_{\mathrm{ML}}, \hat{\sigma}_{\mathrm{ML}}] = \underset{A,B,C,F,\sigma}{\operatorname{argmax}} \ L(A, B, C, f, \sigma | \hat{\mathbf{T}}_{\mathrm{h}}) \tag{4}$$

where $\hat{\mathbf{T}}_{\mathrm{h}}$ denotes the vector of transition level estimators achieved via histogram test:

$$\hat{\mathbf{T}}_{\mathrm{h}} = \left[\hat{T}_{\mathrm{h}}[1] \quad \hat{T}_{\mathrm{h}}[2] \quad \dots \quad \hat{T}_{\mathrm{h}}[2^b - 1]\right]^{\mathrm{T}} \tag{5}$$

Using this solution, the following challenges are to be faced: the accuracy of transition level estimators strongly influences the accuracy of the signal parameters estimators, thus the reliability of the quality measures of the ADC under test. If the transition level estimators show high variance (due to the low number of samples) or are biased (due to incoherent sampling in the measurement record) the inappropriate transition level estimators mislead the approximate ML signal parameter estimators which can be biased this way and do not show better properties than LS estimators. On the other hand performing histogram test with acceptable precision requires long measurement records (e.g. for a 16-bit converter it is not superfluous to use 1 million samples), and fulfilling coherence condition demands very accurate frequency estimation [6].

An other way to decrease the parameter space is to compress the information is provided by the transition levels. If the integral nonlinearity of the converter can be described using a few (e.g. up to 10) parameters without losing essential information, the likelihood function can be optimized with respect to 10..15 parameters which can be numerically treatable. The following sections examine different ways of parameterization to investigate the possibility of solving the maximum likelihood problem via reducing the parameter space.

III. APPROXIMATION OF THE STATIC TRANSFER CHARACTERISTIC USING CHEBYSHEV POLYNOMIALS

Using first-kind Chebyshev polynomials to estimate the integral nonlinearity of the converter is very feasible due to the attractive properties of these functions with sinusoidal arguments [7]:

$$T_k(\cos(\theta)) = \cos(k\theta) \tag{6}$$

where T_n denotes the n^{th} Chebyshev polynomial of the first kind. The nonlinearity of the quantizer can be decomposed to two components: a continuous nonlinearity described by Chebyshev polynomials and an ideal quantizer that provides the step-like quantizer characteristic (see figure 1).

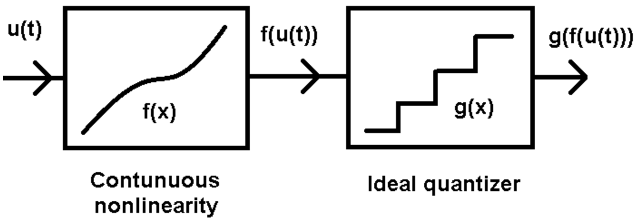

Fig. 1. Decomposing nonlinearity of a nonideal quantizer

Using sinusoidal excitation $u(t) = A\cos(2\pi f t) + C$ the response of the continuous nonlinearity (denoted by $f(x)$) is

$$z(t) = f(u(t)) = f(A\cos(2\pi f t) + C) \tag{7}$$

Since the response of the continuous nonlinearity is a multiharmonic signal, one can use the following equation:

$$z(t) = f(A\cos(2\pi f t) + C) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos(k 2\pi f t) \tag{8}$$

This way using equations 6, 7 and 8 the nonlinearity $f(x)$ can be expressed using Chebyshev polinomials of the first kind:

$$f(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k T_k\left(\frac{x - C}{A}\right) \tag{9}$$

Using equation 9 the parameters describing the nonlinearity of the converter (the coefficients of the Chebyshev polynomials) can be directly calculated using the Fourier series of the recorded signal. The Fourier series can be calculated using multisine wave fit in least squares sense minimizing the following cost function:

$$\sum_{n=1}^{M} \left(y_n - \left(\sum_{k=1}^{N} a_k \cos(k 2\pi f t_n) + \sum_{k=1}^{N} b_k \cos(k 2\pi f t_n) + \frac{a_0}{2} \right) \right)^2 \tag{10}$$

where M denotes the number of samples, N denotes the number of harmonic components estimated and the amplitudes corresponding to the k^{th} harmonic component can be calculated this way:

$$c_k = \sqrt{a_k^2 + b_k^2} \tag{11}$$

Using an initial frequency estimator this model is linear in parameters thus the LS fit can be performed in one step without iterations. Choosing the number of harmonic components to be estimated has key importance in this method: the nonlinearity of the converter is described using $N+1$ parameters via Chebyshev polynomials. Using too large N propounds the risk of fitting the higher harmonics to the samples of noise (if higher harmonics are not large enough compared to the power of noise at that frequency). On the other hand the aim is to reduce the parameter space, thus high number of parameters describing the nonlinearity does not have any advance compared to the unparameterized description of nonlinearity. Naturally using too small N reduces the information about the static transfer characteristic, thus the error of the Chebyshev approximation can be unacceptably large. In the experimental comparison options $N = 5$, $N = 10$ and $N = 15$ are examined. The proposed algorithm is the following:

- Estimate the frequency of the fundamental harmonic using a fast and accurate method (e.g. the one described in [8]).
- Perform a multisine wave fit in least squares sense using the achieved frequency estimator. Choose the number of harmonic components (N) depending on the power of harmonic components and noise (usually between 5 and 15).
- Calculate the coefficients of the Chebyshev polynomials using the amplitudes achieved via multisine wave fit.
- Calculate the approximate values of code transition levels using the Chebyshev approximation and the ideal quantizer:

  $$T_r[k] = \hat{f}(T_i[k]) \tag{12}$$

  where $T_{\mathrm{r}}[k]$ denotes the k^{th} code transition level and $T_i[k]$ denotes the k^{th} transition level of the corresponding ideal quantizer. $f(x)$ is the continuous nonlinearity approximated by the Chebyshev polynomials.

  $$\hat{f}(x) = \frac{c_0}{2} + \sum_{k=1}^{N} c_k T_k\left(\frac{x-C}{A}\right) \tag{13}$$

  Note that $\hat{f}(x)$ is an approximation of $f(x)$ using a finite number of polynomials and

  $$\lim_{N\to\infty} \hat{f}(x) = f(x) \tag{14}$$

- Evaluate the likelihood function using the Chebyshev coefficients instead of the individual transition levels: replace $L(A, B, C, f, \sigma, T[1], T[2], ..., T[2^b-1])$ by $L(A, B, C, f, \sigma, c_0, c_1, ..., c_N)$ as each $T[k]$ depends only on the c_k values.
- Optimize the likelihood function with respect to 10..20 parameters using the robust downhill simplex method (a.k.a. Nelder-Mead mehod) [9]. The extremum provides approximate maximum likelihood estimators for A, B, C, f, σ, c_0, c_1, ..., c_N. Then $\hat{T}[1]$, $\hat{T}[2]$, ..., $\hat{T}[2^b-1]$ can be calculated using $\hat{c}_0$, $\hat{c}_1$, ..., $\hat{c}_N$.

IV. APPROXIMATION OF THE STATIC TRANSFER CHARACTERISTIC USING FOURIER COEFFICIENTS

The information regarding the integral nonlinearity can also be compressed using the Discrete Fourier Transform of the INL vector. The idea is that high-frequency components of the INL only describe the local behavior of the nonlinearity and these local attributions of the transfer characteristic do not influence largely the properties of the quantized signal. The shape of nonlinearity that mainly determines the harmonic components can be described with the first few elements of the Fourier series. The proposed algorithm in this case is the following:

- Perform a histogram test to achieve initial transition level estimators. If conditions of the histogram test are fulfilled poorly use the ideal transition levels.
- Calculate the DFT of the integral nonlinearity (INL) estimators, and discard all values except for the DC component and the first N values (in the experimental comparison N will be set to 2, 5 and 7). The INL will be described using $2N+1$ parameters: d_0 the DC component, d_k and e_k are the real and imaginary parts of the k^{th} element of the DFT respectively.
- Evaluate the likelihood function using the Fourier coefficients instead of the individual transition levels: replace $L(A, B, C, f, \sigma, T[1], T[2], ..., T[2^b-1])$ by $L(A, B, C, f, \sigma, d_0, d_1, ..., d_N, e_1, e_2, ..., e_N)$ due to each $T[k]$ depends only on the d_k and e_k values.
- Optimize the likelihood function with respect to the 10, 16 or 20 parameters using the Nelder-Mead method. The extremum provides approximate maximum likelihood estimators for A, B, C, f, σ, d_0, d_1, ..., d_N, e_1, e_2, ..., e_N. Then $\hat{T}[1]$, $\hat{T}[2]$, ..., $\hat{T}[2^b-1]$ can be calculated using $\hat{d}_0$, $\hat{d}_1$, ..., $\hat{d}_N$, $\hat{e}_1$, $\hat{e}_2$, ..., $\hat{e_N}$.

V. APPROXIMATION OF THE STATIC TRANSFER CHARACTERISTIC USING POLYNOMIAL REGRESSION

The most straightforward way to parameterize the integral nonlinearity is to fit polynomials to the transfer characteristic in least squares sense. The polynomial regression with N coefficients can be performed in the following way:

$$\mathbf{h} = (\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T\mathbf{y} \quad (15)$$

where $\mathbf{h}$ contains the polynomial coefficients

$$\mathbf{h}^T = \begin{bmatrix} h_0 & h_1 & \dots & h_N \end{bmatrix} \quad (16)$$

$\mathbf{D}$ denotes the polynomials of the ideal transition levels:

$$\mathbf{D} = \begin{bmatrix} T_i[1] & T_i[1]^2 & \dots & T_i[1]^{N-1} \\ T_i[2] & T_i[2]^2 & \dots & T_i[2]^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ T_i[2^b-1] & T_i[2^b-1]^2 & \dots & T_i[2^b-1]^{N-1} \end{bmatrix} \quad (17)$$

and $\mathbf{y}$ contains the estimated transition levels:

$$\mathbf{y}^T = \begin{bmatrix} \hat{T}[1] & \hat{T}[2] & \dots & \hat{T}[2^b-1] \end{bmatrix} \quad (18)$$

The proposed algorithm in this case is the following:

- Perform a histogram test to achieve initial transition level estimators.
- Perform polynomial regression of the static transfer characteristic as described previously with a suitable number of coefficients ($N = 5$, $N = 10$ and $N = 15$ will be examined).
- Evaluate the likelihood function using the polynomial coefficients instead of the individual transition levels: replace $L(A, B, C, f, \sigma, T[1], T[2], ..., T[2^b - 1])$ by $L(A, B, C, f, \sigma, h_0, h_1, ..., h_{N-1})$ as each $T[k]$ depends only on the h_k values.
- Optimize the likelihood function with respect to the 10..20 parameters using the Nelder-Mead method. The extremum provides approximate maximum likelihood estimators for A, B, C, f, σ, d_0, d_1, ..., d_N, Then $\hat{T}[1]$, $\hat{T}[2]$, ..., $\hat{T}[2^b - 1]$ can be calculated using $\hat{h}_0$, $\hat{h}_1$, ..., $\hat{h}_{N-1}$.

VI. EXPERIMENTAL COMPARISON OF DIFFERENT METHODS

To compare the parameterization methods we used a 12-bit nonideal quantizer. The transfer characteristic of this quantizer is the measured transfer characteristic of a 12-bit ADC used in the NI-9201 data acquisition board manufactured by National Instruments. The integral nonlinearity has been measured via histogram test using 500000 samples, paying attention to meet the coherence and the Carbone-Chiorboli conditions. This way the the transfer characteristic used in the simulations is modeling the real ADC with very high accuracy. Note that while investigating the quality of different methods for INL approximation the shape of the transfer characteristic to be approximated is not critical, nevertheless it is worth to use a realistic one. The quality of fitting is evaluated from three different aspects:

- the l_2 norm of the INL fitting residuals (see Fig. 2)
- the l_∞ norm of the INL fitting residuals (see Fig. 3)
- the l_2 norm of the difference of two quantized sine waves. One is a full scale sine wave quantized by the original quantizer and the other is the same sine wave quantized by the approximated quantizer (see Fig. 4)

It is also attractive to examine the l_∞ norm of the difference of the two quantized sine waves as a fourth aspect. However, the difference of the quantized signals contains near-zero integer numbers and the l_∞ norm is largely meaningless in this case. In this comparison, the l_∞ norm of the difference was 1 for all the approximation methods. The following figures compare the quality of the approximation using different approaches.

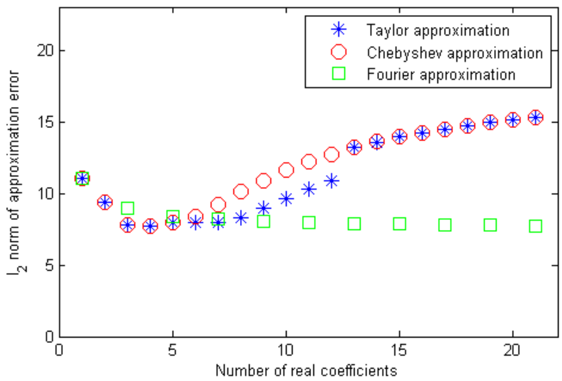

Fig. 2. Comparison of l_2 norms of approximation errors

The number of coefficients means the number of real coefficients for each kind of approximation. In this comparison integer numbers from 1 to 21 have been used as number of coefficients. Regarding Taylor and Chebyshev approximation this quantity is straightforward. Since in Fourier parameterization the coefficients are complex (except for the DC component) the number of real parameters is $2P - 1$ if P denotes the number of used Fourier coefficients (including the DC coefficient). This way the Fourier approximation has been examined for odd numbers between 1 and 21 as number of real coefficients.

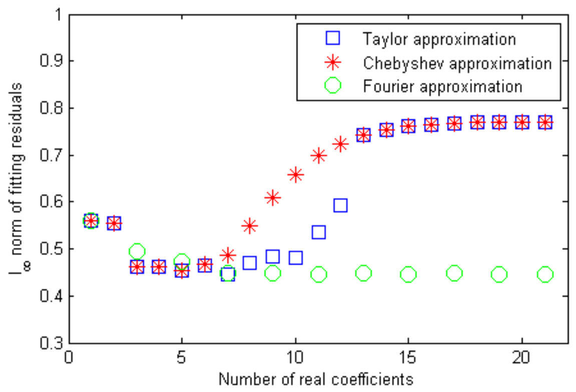

Fig. 3. Comparison of l_∞ norms of approximation errors

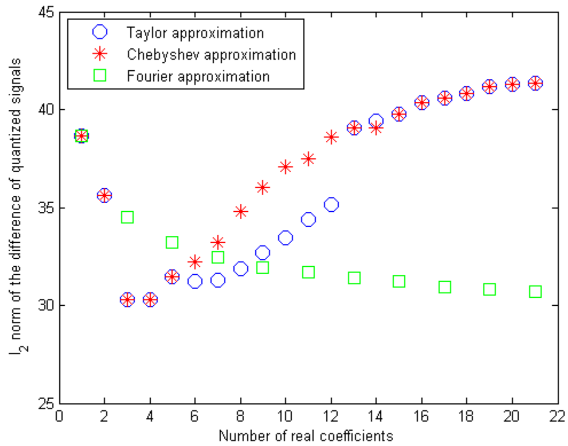

Fig. 4. l_2 norm of the differences of quantized signals using the original and the approximated quantizer

VII. CONCLUSIONS

Parameterization of the static transfer characteristic of ADCs is essential to perform approximate maximum likelihood estimation of signal and ADC parameters without excessively large efforts. The investigations detailed above show that this parameterization can be performed different ways. Taylor and Chebyshev approximation are appropriate for low numbers of coefficients ($P < 5$). For larger numbers numerical problems rise and mislead the approximation. In case of Taylor approximation the condition of the $\mathbf{D}$ matrix becomes very poor when increasing the number of coefficients (see table 1). In case of Chebyshev approximation the numerical optimization of the coefficients lead to potential local extrema (in this comparison a nonlinear least squares fit has been used to optimize the Chebyshev coefficients). For Taylor and Chebyshev approximation the fitting error starts to increase instead of

Table 1. Condition number of matrix $\mathbf{D}$ for Taylor approximation

Number of coefficients	Condition number
3	2.2516e+07
4	1.0396e+11
5	4.6961e+14
6	2.0890e+18

decreasing when the number of coefficients exceeds approximately 5. However the quality of fitting at these low numbers of coeeficients is useful and usually provide better fit than Fourier approximation (see Fig. 4). Fourier approximation provides robust fit without numerical challenges. The quality of fitting increases with the number of coefficients in a monotonic way. In conclusion, both of these approximation methods can be used to reduce the parameter space and to make approximate maximum likelihood estimation of ADC and signal parameters possible. In the future the best of these methods shall be built in the ML estimation algorithms to provide reliable approximate ML estimators for analog signal and ADC parameters with acceptable computational demands and without unsolvable numeric problems.

VIII. ACKNOWLEDGMENT

This work has been supported by the National Research, Development and Innovation Office (NKFIH) Grant No. K115820.

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