A Filtered Reference – Filtered Error LMS Algorithm

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Abstract

Adaptive filters updated by the least mean square (LMS) algorithm are succesfully implemented in digital control systems. They are utilized for both plant identification and control purposes. In control appications (e.g. in active noise control) the output of the adaptive filter drives the plant input, and the error signal is derived only at the output of the plant. In such cases the filtered reference LMS algorithm guarantees the stable adaptation. However, the convergence of this algorithm can be very slow, depending on the plant in the control loop. The paper introduces a novel algorithm which filters both the reference and the error signal by the same filter which is designed to provide a loop gain close to the unity along the whole frequency axis. The paper illustrates the efficiency of the proposed method by examples.

1. Introduction

The least mean square (LMS) algorithm is well-known for engineers involved in digital signal processing. It is proved to be a robust algorithm for adaptation of transversal digital filters. In the last two decades many papers and books were published dealing with this topic. A very good introduction to LMS adaptive filters is available in [7] written by the inventor of the algorithm. The structure was applied for adaptive modeling and inverse adaptive modeling, deconvolution, adaptive interference canceling, etc. [7], [8], [4].

In control applications the output of the adaptive filter drives the plant input, and the error signal is derived only at the output of the plant. In such cases the simple LMS algorithm is unstable due to the phase shift caused by the plant. The problem is analyzed e.g. in [5] and the solution is the so-called filtered reference or filtered-X LMS (XLMS) algorithm [7]. This algorithm requires a model of the plant, which the reference signal is filtered by. The identification of the plant can be done off-line, using the simple LMS algorithm. The model shall be so accurate that its phase error does not exceed $\pi/2$, otherwise the adaptive system is unstable. The XLMS algorithm was extended also for multiple input – multiple output controllers [3], and it was successfully applied e.g. for active noise control (ANC) [2].

Although the XLMS algorithm is stable, its convergence can be very slow, depending on the plant in the control loop. In ANC experiments, the suppression of some sinusoidals by the XLMS algorithm requires tens of seconds. This means that the XLMS algorithm is practically unusable in such situations. The origin of this phenomenon is the high dynamics in the acoustic transfer function which plays the role of the plant in ANC systems. Recognizing this drawback of the XLMS algorithm, recently the filtered error or filtered-ε (ELMS) algorithm is introduced [8]. The idea of the algorithm is straightforward: was the inverse of the plant applied in the error path, the structure was identical with the original LMS algorithm and hence it could be fast enough. Unfortunately, the exact inverse of the plant generally can not be applied, due to its non minimum-phase feature. Instead of the exact inverse, the ELMS algorithm utilizes the so-called delayed inverse in the error path. The overall transfer function of the delayed inverse and the plant is approximately a delay, instead of the unity as it is required for the common inverse. It can be proved that this delayed inverse can be constructed for any trasfer function. The reference signal for the ELMS algorithm is delayed according to the delay of the delayed inverse. The identification of the delayed inverse can be done similarly to that of the plant model, using the LMS algorithm.

The ELMS algorithm has excellent convergence properties, but it is not wide-spread in practical applications yet. This paper introduces an alternative structure which also can improve the convergence speed of the XLMS algorithm and offers some advantages compared to the ELMS algorithm. The disadvantageous convergence properties of the XLMS algorithm originate from the high dynamics in the magnitude response of the plant. The convergence rate

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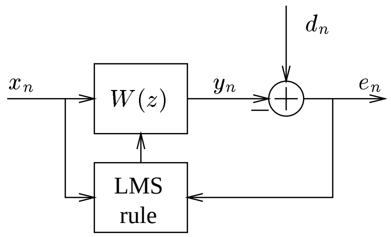

Figure 1. The LMS adaptive filter

of the adaptation depends on the loop gain in the adaptation path, which is proportional to the square of the plant's magnitude response, due to the LMS adaptation rule. The proposed algorithm is a modification of the original XLMS algorithm: the reference signal is filtered not only by the model of the plant, but with an auxiliary filter which is designed to provide an overall magnitude response close to the unity along the whole frequency axis. In order to keep the stability, the error signal has to be also filtered by this auxiliary filter. The paper introduces this algorithm and investigates its behavior.

Section 2. recalls the LMS algorithm and its extensions, and section 3. introduces the proposed structure. Section 4. investigates the main features of the novel algorithm showing examples. The paper is closed with a short conclusion.

2. The LMS Algorithm and its Extensions

2.1. The LMS algorithm

The LMS adaptive filter can be seen in Fig. 1. In this figure $W(z)$ denotes the adaptive transversal filter, x_n, y_n and e_n are the reference signal, the output of the filter and the error signal at time step n, respectively. d_n is the desired signal, which x_n has to be correlated with. The system is described by the following equations:

$$y_n = \mathbf{w}_n^T \mathbf{x}_n \tag{1}$$

$$e_n = d_n - y_n \tag{2}$$

where $\mathbf{w}_n$ denotes the coefficients of the adaptive filter and $\mathbf{x}_n$ is the vector formed from the actual and delayed samples of the reference signal at time step n. The LMS adaptation rule is the following:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu e_n \mathbf{x}_n \tag{3}$$

where the overbar denotes the complex conjugate operator and μ is a positive constant which controls the stability and the convergence rate of the adaptation. The correlation between x_n and d_n can be represented by a discrete transfer function. After a successful adaptation, $W(z)$ approximates this transfer function in a least mean square sense. If

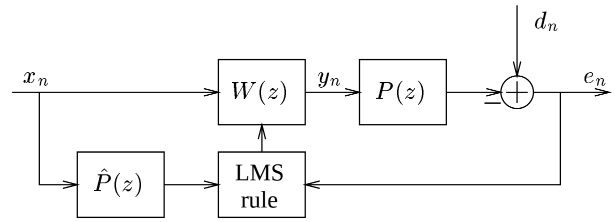

Figure 2. The filtered-X LMS algorithm

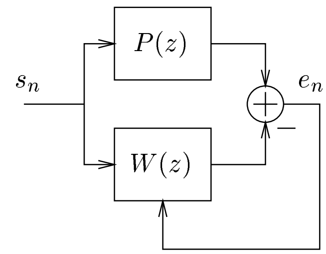

Figure 3. Model identification for the XLMS algorithm

x_n and d_n are input and output signals of a plant, the LMS adaptive filter is able to identify the transfer function of the plant.

2.2. The XLMS algorithm

In control applications the adaptive filter is the controller. In this case $W(z)$ is updated by the XLMS algorithm. The structure can be seen in Fig. 2, where the plant is denoted by $P(z)$. $\hat{P}(z)$ is a model of the plant which is identified off-line. The system is described as follows:

$$e_n = d_n - P(z) y_n \tag{4}$$

where y_n is defined as in (1). (3) is modified in the following way:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu e_n \bar{\mathbf{r}}_n \tag{5}$$

where $\mathbf{r}_n$ is the vector formed from the actual and delayed samples of the filtered reference signal r_n:

$$r_n = \hat{P}(z) x_n \tag{6}$$

$\hat{P}(z)$ can be either infinite or finite impulse response (IIR or FIR) filter, but it is usually an FIR filter. The identification of $P(z)$ can be done by the system depicted in Fig. 3. It is a utilization of the simple LMS adaptive filter. If the excitation s_n is white noise, $W(z)$ provides a satisfactory model of $P(z)$. The system is stable if the phase error of the model does not exceed $\pi/2$ [5], [8].

2.3. The ELMS algorithm

In order to improve the convergence rate of the XLMS algorithm recently the ELMS algorithm is proposed [8].

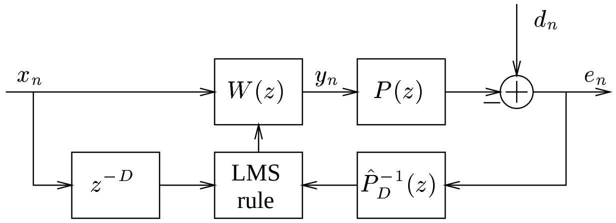

Figure 4. The filtered-ε LMS algorithm

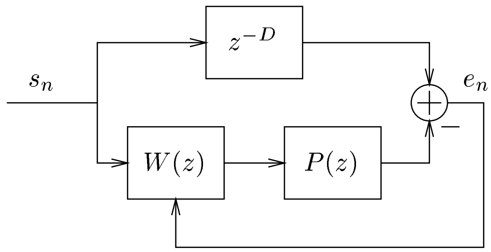

Figure 5. Model identification for the ELMS algorithm

The arrangement can be seen in Fig. 4. The expression $\hat{P}_D^{-1}(z)$ is the so-called delayed inverse of $P(z)$. The system is described by (1) and (4), while (5) is modified in the following way:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu \hat{P}_D^{-1}(z) e_n \bar{\mathbf{r}}_n \tag{7}$$

where

$$r_n = z^{-D} x_n \tag{8}$$

The delayed inverse $\hat{P}_D^{-1}(z)$ can be defined as follows:

$$P(z)\hat{P}_D^{-1}(z) \approx z^{-D} \tag{9}$$

$\hat{P}_D^{-1}(z)$ is an FIR filter and the overall transfer function only approximates the delay. The ideal filter would be a non-causal IIR filter [8]. On the other hand, the meaning of the delayed inverse is obvious assuming Fig. 5. The identification can be done using the simple LMS algorithm as in the case of the XLMS algorithm, only the order of the blocks is different. In this case s_n has to be white noise and the delay has to be set experimentally. The delay applied in the ELMS algorithm is the same as that in the identification structure.

Another problem is the required number of the coefficients of the FIR filter realizing the impulse response of the delayed inverse. Assuming $P(z)$ is rational, the ideal inverse can be constructed by the exchange of the poles and zeros of $P(z)$. If $P(z)$ is minimum phase, the exact inverse can be constructed and no delay is required ($D = 0$). The impulse response is an exponential settling towards the zero and its length can be estimated by the zero of $P(z)$ closest to the unit circle. If $P(z)$ has only non minimum phase zeros, only the delayed inverse exists ($D \neq 0$). The ideal impulse response is non-causal and similar to the previous one for $n = 0, -1, -2, \ldots$. To construct the delayed inverse, this impulse response has to be truncated at time step $n = -D$ and it has to be shifted by D. The length of the inverse filter is D, and it is determined by the zero of $P(z)$ closest to the unit circle as in the previous case. Generally, if $P(z)$ has zeros both inside and outside the unit circle, the impulse response of the delayed inverse consists of exponentially increasing and decreasing parts. Assuming the same goodness for the minimum phase and the non minimum phase zeros, the required length of the delayed inverse is approximately $2D$.

3. The Proposed Filtered Reference – Filtered Error LMS Algorithm

The disadvantageous convergence properties of the XLMS algorithm originate from the high dynamics in the magnitude response of $P(z)$. The convergence rate of the adaptation depends on the loop gain in the adaptation path, which is proportional to $|P(z)|^2$, due to the XLMS adaptation rule (5). The convergence speed of the LMS algorithm is controlled by the parameter μ, which is limited due to the maximum of $|P(z)|$. If $|P(z)|$ has high dynamics, there are some frequency bands, where the loop gain is very small. For any signal appearing in this frequency range the convergence rate will be small.

The magnitude response of the delay in (9) is unity, therefore the ELMS algorithm mentioned above is a solution of the problem. The crutial element in the structure is the delayed inverse. Both the XLMS and the ELMS algorithms need some experiments to set the appropriate length of the model (or the inverse model) filter. However, the ELMS algorithm needs further experiments to set the optimal value of the delay D, which makes the modeling uncomfortable in some cases. The length of the filter in the XLMS algorithm can be estimated by the pole of $P(z)$ closest to the unit circle. Assuming nearly the same goodness of the poles and the zeros of $P(z)$, the delayed inverse in the ELMS algorithm is approximately twice longer than the filter in the XLMS algorithm. In some applications, where hundreds of coefficients are required, this is a meaningful difference.

The proposed filtered reference – filtered error LMS (EXLMS) algorithm tries to solve the problem caused by the dynamics of $|P(z)|$ keeping the simpler filter $\hat{P}(z)$. An FIR filter is applied, which filters both the error and the reference signal. This filter is designed so that the resulted magnitude response oscillates around the unity. The proposed structure can be seen in Fig. 6. The system is a modification of the XLMS structure. The new element in the figure is $H(z)$, which is the filter mentioned above. The

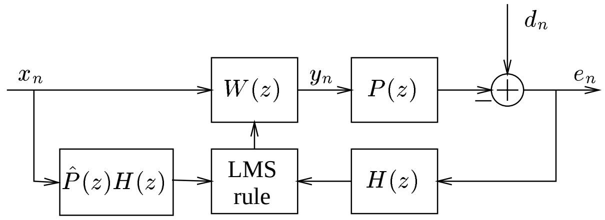

Figure 6. The proposed filtered-X filtered-ε LMS algorithm

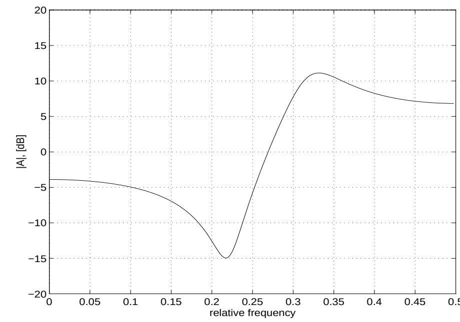

Figure 7. Magnitude response of $P(z)$ in (14)

system is described by (1) and (4), while (5) is modified in the following way:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu H(z) e_n \bar{\mathbf{r}}_n \tag{10}$$

where

$$r_n = H(z)\hat{P}(z)x_n \tag{11}$$

Since $H(z)$ is applied in both pathes, the system is stable. In the following the main assumptions for filter design are introduced.

While the delayed inverse ensures the stability of the adaptation and compensates the dynamics of the plant simultaneously, the proposed method separates these functions. $\hat{P}(z)$ cares of the stability, while $H(z)$ is the compensator, therefore its magnitude response is specified so, that:

$$|H(z)| \approx \frac{1}{|\hat{P}(z)|} \tag{12}$$

The error of the approximation could be higher than it is usual in filter design. It involves that the required number of the coefficients of $H(z)$ can be much lower than that of $\hat{P}(z)$.

The magnitude response of $H(z)$ can be prescribed by the loop gain of the adaptation:

$$L(z) = \mu |H(z)P(z)|^2 \tag{13}$$

where it is assumed that $\hat{P}(z) = P(z)$. In pratical applications the convergence parameter μ is set experimentally to achieve the best convergence rate. In such cases smaller steps than 6dB to change μ have no meaningful influence to the convergence rate. Since μ and the square of the resultant transfer function of the filters control directly the loop gain, $H(z)$ has to be designed so that the resultant magnitude response $|H(z)\hat{P}(z)|$ varies in a 3dB range. The filter design itself can be done using the simple frequency sampling method, where $1/|\hat{P}(z)|$ is sampled. If $H(z)$ is designed so that the resultant magnitude response is "too smooth", i.e. it ripples very close to the unity, the convergence is slower, because of the large delay of $H(z)$. Mathematical proof of the above condition is not available yet, but simulation results prove the statement.

The role of $H(z)$ can be interpreted by the frequency domain adaptive filtering [1]. Frequency domain adaptation provides the possibility to set the convergence parameters at each channel independently, according to the power of the signal appearing in the channel. The idea was successfully applied also for periodic noise control [6]. The proposed method ensures similar possibility, but in the time domain. The following section illustrates the efficiency of the method.

4. Examples

In this section some simulation results are discussed. The plant in all examples is a simple second-order IIR filter:

$$P(z) = \frac{z^2 - 0.4164z + 1.2346}{z^2 + 0.6627z + 0.6414} \tag{14}$$

Its magnitude response can be seen in Fig. 7. The zeros of $P(z)$ lie outside the unit circle. Although it is a very simple transfer function with only about 25dB dynamics, it can illustrate the efficiency of the proposed method in a convincing manner. In all examples the adaptive filter has 200 coefficients and the desired signal d_n is defined as:

$$d_n = x_{n-100} \tag{15}$$

where x_n is the reference signal which is a white noise with the same distribution in all case. The convergence parameter μ is set in all examples experimentally to achive the highest convergence rate.

First the adative filter is updated by the XLMS algorithm (Fig.2). In this case $\mu = 0.0002$, and it is the best one. The error signal can be seen in Fig.8. The convergence is very slow, after 10,000 steps the error signal is about one third of the initial value.

From now on the proposed EXLMS algorithm (Fig.6) is simulated. In the first experiment $H(z)$ has 11 coefficients. The magnitude response of $P(z)$ is slightly compensated as it can be seen in Fig.9. In this case $\mu = 0.001$, and the error signal can be seen in Fig.10. The amplitude of

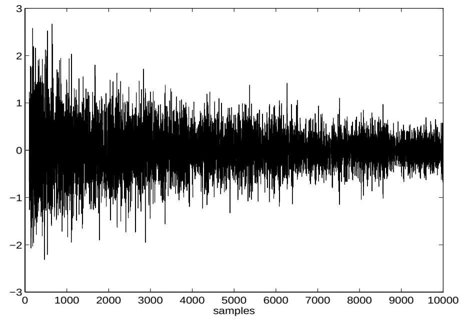

Figure 8. Error signal of the XLMS algorithm

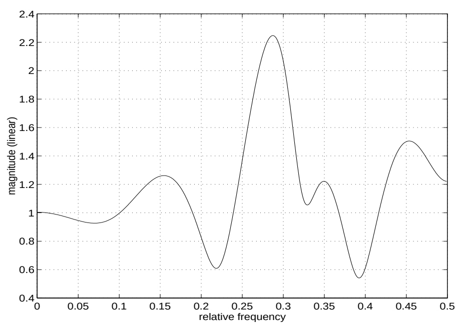

Figure 9. $|H(z)P(z)|$ with 11 coefficients of $H(z)$

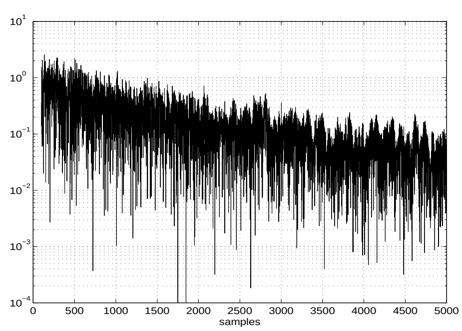

Figure 10. Error signal of the EXLMS algorithm with 11 coefficients of $H(z)$

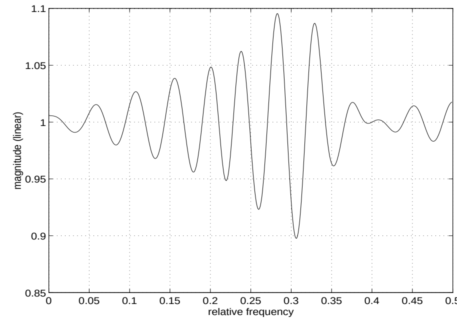

Figure 11. $|H(z)P(z)|$ with 41 coefficients of $H(z)$

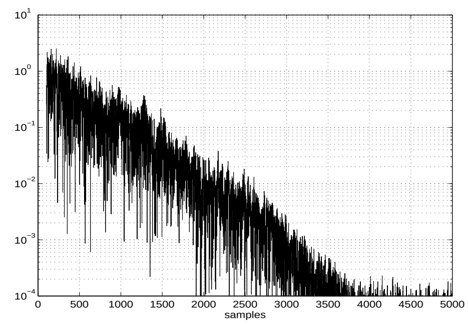

Figure 12. Error signal of the EXLMS algorithm with 41 coefficients of $H(z)$

the error signal is plotted in logarithmic scale. After 5,000 steps the error signal is more than 10 times smaller than the initial value, it means that $H(z)$ with only 11 coefficients could improve the convergence rate.

In the second experiment with the EXLMS algorithm $H(z)$ has 41 coefficients. The magnitude response of $P(z)$ is well compensated as it can be seen in Fig.11. In this case $|H(z)P(z)|$ varies approximately in a 3dB range, and $\mu = 0.005$, the error signal can be seen in Fig.12. The amplitude of the error signal is plotted in logarithmic scale. After 4,000 steps the error signal is about 10,000 times smaller than the initial value, it means that $H(z)$ with 41 coefficients could significantly improve the convergence rate. Since the adaptation can be treated as complete, the coefficients of the adaptive filter $W(z)$ are set. Since d_n is a delayed version of x_n, the adaptive filter is set to the impulse response of the delayed inverse of $P(z)$. It can be seen in Fig.13.

In the third experiment with the EXLMS algorithm $H(z)$ has already 101 coefficients. The magnitude response of $P(z)$ is overcompensated as it can be seen in Fig.14. In this case $\mu = 0.005$, and the error signal can be seen in Fig.15. Although the overall magnitude response is very smooth, the convergence rate a bit smaller than it was

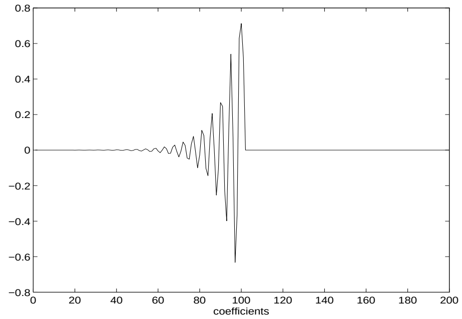

Figure 13. The delayed inverse of $P(z)$

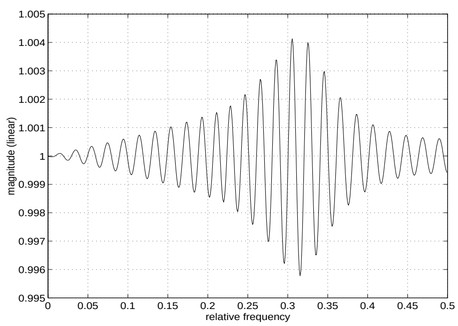

Figure 14. $|H(z)P(z)|$ with 101 coefficients of $H(z)$

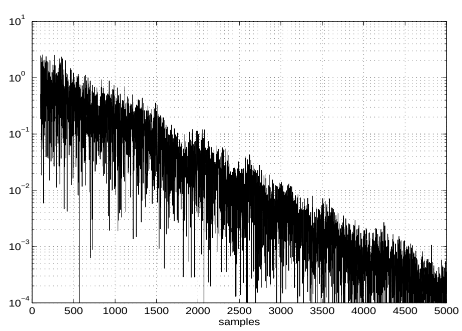

Figure 15. Error signal of the EXLMS algorithm with 101 coefficients of $H(z)$

in the previous case. This example illustrates that there is no reason to approximate the unity magnitude response with a very small error.

5. Conclusion

The paper presented a new filtered reference – filtered error LMS algorithm. This algorithm is proposed to utilize in control applications where the adaptive filter is the controller and the plant magnitude response has high dynamics. The proposed method in such cases provides much higher convergence rate than the widely used filtered reference LMS algorithm and it can be a real alternative for the recently introduced filtered error LMS algorithm.

References

[1] J. E. R. Ferrara. Frequency domain adaptive filtering. In C. F. N. Cowan and P. M. Grant, editors, *Adaptive Filters*. Prentice-Hall, Inc., 1985.

[2] S. J. Elliot and P. A. Nelson. Active noise control. *IEEE Signal Processing Magazine*, 10(4):12–35, Oct. 1993.

[3] S. J. Elliot, I. M. Stothers, and P. A. Nelson. A multiple error LMS algorithm and its application to the active control of sound and vibration. *IEEE Trans. Acoust., Speech, Signal Processing*, ASSP-35(10):1423–1434, Oct. 1987.

[4] J. R. Glover. Adaptive noise canceling applied to sinusoidal interferences. *IEEE Trans. Acoust., Speech, Signal Processing*, ASSP-25(12):484–491, Dec. 1977.

[5] D. R. Morgan. An analysis of multiple correlation cancelation loops with a filter in the auxiliary path. *IEEE Trans. Acoust., Speech, Signal Processing*, ASSP-28(8):454–467, Aug. 1980.

[6] L. Sujbert and G. Péceli. Periodic noise cancelation using resonator based controller. In *1997 Int. Symp. on Active Control of Sound and Vibration, ACTIVE '97*, pages 905–916, Budapest, Hungary, Aug. 1997.

[7] B. Widrow and S. D. Stearns. *Adaptive Signal Processing*. Prentice-Hall, Inc., 1985.

[8] B. Widrow and E. Walach. *Adaptive Inverse Control*. Prentice-Hall, Inc., 1996.