# A COMPARISON OF LEAST SQUARES AND MAXIMUM LIKELIHOOD METHODS BASED ON SINE FITTING IN ADC TESTING

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Abstract: ADC test methods require the best possible reconstruction of the input signal of the ADC under test from the acquired, therefore erroneous, ADC output data. The commonly used least squares (LS) fit and the recently introduced maximum likelihood (ML) estimation are competing methods. This paper presents a simulation-based comparative study of these estimation methods with the goal to investigate the behaviour of both methods and to determine their limits. Two alternative algorithms for the calculation of the maximum likelihood fit are considered (gradient-based minimization and differential evolution). The main finding is that while for low-INL (linear) ADCs the two methods (LS and ML) give similar results, for practical (almost always nonlinear) ADCs ML is definitely better.

**Keywords:** ADC test, maximum likelihood estimation, least squares method, LS method, four-parameter fit, signal recovery, estimation of signal parameters, gradient-based method, differential evolution.

## 1. INTRODUCTION

Standardized dynamic test methods for analog-to-digital converters (ADCs) ([1]–[3]) are based on comparison of acquired ADC output codes with the ADC input stimulus. The stimulus is not exactly known and cannot be measured with the necessary accuracy, therefore it must be reconstructed from the erroneous ADC output codes acquired during testing. Any inaccuracy in the estimation of ADC stimulus parameters leads to inaccuracy in determination of ADC parameters measured by the dynamic test.

The most common way how to recover input signal and estimate its parameters is least squares (LS) fitting. This is also recommended in the standards. According to the theory, LS fitting gives the best estimation under the condition that the observation (quantization) noise is additive to the input, is independent, white, and normally distributed with zero mean ([4], [5]). This all is clearly not true for ADC testing ([13], [14]) and therefore the LS fit is usually worse than ML estimation would be.

The general, systematic "best" way of estimation is fitting based on the maximum likelihood (ML) method. This idea was introduced in [6] for sine wave and later it was generalized also for exponential stimulus ([7]). It was used also for estimation of ADC noise in [14]. The ML estimation is optimal in a certain sense, but has disadvantages like its computation complexity and possible local minima.

Although both fitting procedures are known, until now no deeper comparative study on limitations of these methods has been performed. The main novelty of this paper is just this comparative research. Moreover, we also examined the differential evolution (DE) based optimization method for ML fit [9]. Minimization of the same cost function can be implemented in different software environments (Matlab and LabVIEW), and with different minimization strategies (e.g. gradient-based or differential evolution). We have solved minimization in Matlab with the Levenberg-Marquardt method, and in LabVIEW with the DE method.

## 2. FITTING METHODS

The general setup for dynamic ADC testing is shown in Fig. 1. To perform simulation experiments we developed a few software modules including non-ideal ADC model with optional test specific INL. The modules enable simulating real ADC test according to Fig. 1.



Fig. 1. General setup for dynamic ADC testing

#### 2.1. Least Squares (LS) fit

LS fitting is a well known and very often used procedure for recovering distorted and noisy signals in testing and measurements [1]. Estimation of parameters of ADC input signal is obtained from minimization of the cost function  $CF_{LS}$ :

$$\min_{\mathbf{a}} \left( \operatorname{CF}_{LS}(\mathbf{a}) \right) = \min_{\mathbf{a}} \sum_{n=0}^{N-1} \left( y(n) - f(\mathbf{a}, n) \right)^2$$
(1)

where  $f(\mathbf{a},n)$  is the time model of the test signal, **a** is a vector of its unknown parameters, and y(n) are digitized samples of test signal applied to the input of ADC under test, taken at sampling instances  $t_n$ . Solution of (1) leads to a system of equations which is nonlinear in the cases of the four-parameter fit of a sine wave or for dual slope exponential stimulus. Solution of such a nonlinear system requires application of an appropriate numerical method, increasing the complexity of the LS method.

#### 2.2. Maximum likelihood method

Maximum likelihood method comes from estimation theory and looks for the most probable ADC input signal. ML fitting is based on maximization of a likelihood function  $L(\mathbf{a})$ :

$$\max_{\mathbf{a},\,\mathbf{q},\,\sigma} \left( L(\mathbf{a}) \right) = \max_{\mathbf{a},\,\mathbf{q},\,\sigma} \prod_{n=0}^{N-1} P(y(n) = Y_{\mathbf{a},\,\mathbf{q},\,\sigma}(n)), \tag{2}$$

where **q** is the vector of ADC transition code levels,  $\sigma$  is the standard deviation of the Gaussian noise of the input electronics, assuming here that the noise samples are independent. P(y(n) = Y(n)) is the probability that given the signal model, the *n*-th ADC output sample y(n) is equal to the measured output value Y(n), which represents a possible output code. To simplify the maximization  $L(\mathbf{a})$ , minimization of the sum of negative logarithms of probabilities P(.) is preferred.

$$\arg \max_{\mathbf{a}, \mathbf{q}, \sigma} (L(\mathbf{a})) \approx \arg \min_{\mathbf{a}, \mathbf{q}, \sigma} (-\ln(L(\mathbf{a}))) =$$

$$= \arg \min_{\mathbf{a}, \mathbf{q}, \sigma} \left( -\sum_{n=0}^{N-1} \ln P(y(n) = Y_{\mathbf{a}, \mathbf{q}, \sigma}(n)) \right)$$
(3)

Searching the extreme value can be a complex task and it can be performed only by an appropriate numerical method.

#### 2.3. Implementation of fitting and comparison

LS fit for a sine wave was implemented according to standards ([1]-[3]) in the form of 4-parameter fit ([11]. When the ADC was overloaded, LS fit was made both to the full record as described in the standard, and/or to the record with the overloaded samples discarded, in order to decrease bias.

The ML algorithm is more complex, since for proper estimation the transition levels need also to be estimated. Since for a 8-bit ADC 260-parameter minimization is needed [6]-[7], this is not practical with full attempt. We chose to determine the levels T[k] from the cumulative histogram. This provides values which are close to the best (and thus to the ML) estimates.

For the following, T[k] denotes the comparison level between codes k-l and k, with  $k = 1, 2, ..., 2^{b}$ -1 (for input level T[k], probabilities of the output codes k-l and k are equal to 50%. respectively).

The possible output codes are  $0,1,...,2^{b}-1$ . *NCH[k]* (for each code) denotes the normalized cumulative histogram: the sum of the relative frequencies for all codes  $c \le k$ . Thus, *NCH*[ $2^{b}-1$ ]=1.

We know that the cumulative histogram contains information only about the T[k] values which are excited. This means that with a sine wave of carefully chosen (almost full-range, but not overloaded) amplitude we can estimate almost all T[k]values – at least those which are excited in this experiment. As "calibration step" we set the two extreme but still excited T[k]-s ( $T[k_1]$  and  $T[k_2]$ ) to the corresponding nominal values (*k*-0.5). For the not measured T[k]-s (the levels outside the above values) the nominal values are set: (*k*-0.5).

For  $k_1 < k < k_2$ , assuming that the noise is small, the amplitude and dc level of the sine wave can be estimated from the cumulative histogram values for the above extreme T[k]-s [10]. This can be done by a simple manipulation on the CDF (Cumulative Density Function):

$$F(x) = \frac{1}{\pi} \arcsin\left(\frac{x-\mu}{A}\right) + 0.5$$
(4)

Since  $NCH(k_i-1)$  gives the probability (F(x)) that  $x < T[k_i]$ , the equations to be fulfilled for  $\mu$  and A are

$$A\sin(\pi(NCH(k_1-1)-0.5)) = -A\cos(\pi \cdot NCH(k_1-1)) = T(k_1) - \mu \text{ and } -A\cos(\pi \cdot NCH(k_2-1)) = T(k_2) - \mu$$
(5)

with  $T[k_i] = k_i - 0.5$ , from which

μ

$$A = \frac{k_2 - k_1}{\cos(\pi \cdot NCH(k_1 - 1)) - \cos(\pi \cdot NCH(k_2 - 1))}$$
(6)  
=  $\frac{(k_2 - 0.5)\cos(\pi \cdot NCH(k_1 - 1)) - (k_1 - 0.5)\cos(\pi \cdot NCH(k_2 - 1))}{\cos(\pi \cdot NCH(k_1 - 1)) - \cos(\pi \cdot NCH(k_2 - 1))}$ (7)  
 $\frac{k_2\cos(\pi \cdot NCH(k_1 - 1)) - k_1\cos(\pi \cdot NCH(k_2 - 1))}{\cos(\pi \cdot NCH(k_1 - 1)) - \cos(\pi \cdot NCH(k_2 - 1))} - 0.5$ 

Thus we have the parameters for the input CDF, and its inverse can be applied to have the T[k]-s without the bias due to the non-uniform distribution of the sine wave:

$$T(k) = \mu - A\cos(\pi \cdot NCH(k-1)), k = k_1 + 1, \dots, k_2 - 1$$
(8)

Now the log-likelihood function  $CF = \sum \log(P_n)$  can be maximized via the 5 parameters  $(A, B, C, f_0, \sigma)$ .

We have implemented this minimization in two ways (gradient-based method: Levenberg-Marquardt in Matlab, and the DE method: GlobalOptimization.vi in LabVIEW). The results were approximately the same.

DE methods approach the optimum by mutating and improving the candidate parameters from the initial ones. Fig. 2 illustrates how the algorithm works. GlobalOptimalization.vi uses (if available) multi-core processing, which speeds up the calculation of the best parameters.



Fig. 2. Algorithm of DE method

For comparison of the accuracies of signal recovery achieved by LS and ML methods we determined two parameters:

- 1. Estimation error D defined as RMS value of the difference between estimated signal and the exact input signal.
- Difference between SINADs calculated from LS fit, or from ML fit and the value calculated from exactly known ADC stimulus.

These values can be evaluated in simulation only, because to do this, we need the exact excitations.

The RMS value was chosen because it is the basic descriptor of the error which has influence on nearly all final ADC dynamic parameters such as ENOB and SINAD. Moreover, a simple comparison of differences between, e.g., sine wave amplitudes is not well readable from the point view of its influence on the final error in calculation of ADC parameters. The estimation error  $D_{\text{method}}$  (RMS value of the difference between the recovered signal by a given method and the known exact one) was calculated according to the formula:

$$D_{\text{method}} = \sqrt{\frac{1}{N} \sum_{n=0}^{N-1} (y_{\text{method}}(n) - y_{\text{exact}}(n))^2}, \qquad (9)$$

where  $y_{\text{method}}(n)$  are the samples calculated from signal recovered by the given method and  $y_{\text{exact}}(n)$  are the equivalent samples of the exact ADC stimulus.

Comparison of SINADs calculated from LS and ML fits to SINAD calculated from the exact value of stimulus was calculated only in tests, where the stimulating sine wave did not overload ADC full scale as it is required by standards. SINAD in dB was calculated according to the standard formula defined in [1]:

$$SINAD_{method} = 20\log \frac{A_{method}}{NAD_{method}},$$
 (10)

where  $A_{\text{method}}$  is RMS value of sine wave recovered by a given method or from the exactly known ADC stimulus and NAD<sub>method</sub> is rms value of noise and distortion given by:

$$NAD_{\rm method} = \sqrt{\frac{1}{N} \sum_{n=0}^{N-1} (y_{\rm record}(n) - y_{\rm method}(n))^2} , \qquad (11)$$

where  $y_{record}(n)$  are digitised samples recorded on ADC output. Note that the influence of *NAD* is not very significant in SINAD if it is much smaller than standard deviation of the quantization noise, 0.3 LSB (LSB is the least significant bit: the resolution of the ADC: range/2<sup>b</sup> with *b* the number of bits), but more significantly contributes to ENOB.

### 3. SIMULATION RESULTS

## 3.1 LS vs. ML estimation

The first experiments were performed on linear ideal ADCs with different resolutions (up to 16 bits), and standard deviation of noise equal to 0.2 LSB, to represent ADC code alternating behaviour. One would expect that ideal (linear) ADC characteristic causes similar LS and ML errors. Fig. 3 shows a zoomed segment of record of digitized stimulus from the ideal 8-bits ADC together with exact stimulus, and recovered signals. Stimulus did not overload ADC full scale. The differences between both recovered signals and exact stimulus are hardly visible.



Fig. 3. Comparison of LS and ML fit, digitized and exact stimulus (zoomed time segment) for ideal ADC characteristics

The difference between the LS and ML fits is visible on the example of a nonlinear ADC. The following tests were focused on effects on ADC nonlinearity at recovering the test signals by LS and ML fits.

To approximate as much as possible a real ADC in simulations, the INL measured on the real ADC in USB 6009 by National Instruments was implemented in the simulation software. The real INL was simplified by rounding the lowest bits, in order to be equivalent to INL of a 8 bits ADC as it is shown in Fig 4. Approximate maximum likelihood estimates of the transition levels were determined from the sample record via the histogram method [1], with the standard correction for the sine wave, as described in Section 2.3. In our view, this is the maximum information that can be extracted from the record for the T[k] values. During the ML fit, these estimates were kept constant, and the other 5 parameters were optimized.



Fig. 4. Nonlinear INL of a real ADC used in the following simulations



Fig. 5. Comparison of LS and ML fits, with digitized and exact stimulus (zoomed time segment) for nonlinear ADC characteristics. The result of ML estimation is much closer to the true signal than that of the LS fit.

We have investigated the behaviour of the fits for a nonlinear 8-bit ADC and 500 coherently taken samples with standard deviation of noise  $\sigma = 0.2$  LSB for varying ADC overloading. If the input signal covers ADC range up to 100 percent, both methods give small errors, but ML errors are smaller than LS.

The RMS value of the difference of the LS fit from the exact stimulus quickly increased if the input stimulus even only slightly overloaded the ADC input range and if LS fit was applied on the whole record, therefore we discarded the overloaded samples in LS fits. Fig. 6 shows the fitting errors for nonlinear ADC. The estimation errors for both fits are small, but the ML fit having smaller error.



Fig. 6. LS vs. ML fit for overloaded ADC (100% is equal to ADC full scale). LS fit uses pre-processed record.

The resulting practical conclusions coming from the performed experiments is that

- the dominating error factor in "blind" LS fitting is ADC overloading (if this happens). ML fit is much better for any, even very small, ADC overloading and it does not require to pre-process the record and eliminate overloading samples. LS fit can give nearly the same results if the overloaded samples are excluded from the record,
- o even with pre-processed data used for LS fit, the error of the ML fit is somewhat smaller than that of the LS fit (Fig. 6).

## 3.2 A study of the error of the recovered stimulus

In all the simulations below, we used the nonlinear ADC characteristic as in Fig 4.

The next study was focused on the accuracy of recovered stimulus for different resolutions of the ADC. The test conditions were that the ADC under test was just not overloaded,  $\sigma = 0.2$  LSB and the length of record was  $N = 2^{(b+2)}$ , where *b* is the ADC resolution in bits. The results are the mean values of *D* with the standard deviations estimated from 20 experiments.

The estimation errors (Eq. 8) are shown in Fig. 7.



Fig. 7. Dependence of estimation error on number of ADC bits for LS and ML fit

The achieved results show that the error of the ML fit is definitely smaller than that of the LS fit.

The next experiment was focused on dependence of estimation error *D* on number of samples in the record with coherent sampling. Fig. 8 shows this dependence for 8-bits ADC and  $\sigma = 0.2$  LSB. The errors of the ML fit are definitely smaller.



Fig. 8. Dependence of estimation error on number of samples for 8 bit ADC for LS and ML fit.

The following experiments were focused on the influence of the ADC noise that causes alternation of ADC output codes. The results achieved for 8 bits ADC and 2048 samples in record are shown in Fig. 9. According to these simulations, the ML fit has smaller error.



Fig. 9. Dependence of LS and ML fit error on noise level.

The next study examines the effects of different fitting methods on precision of calculation of SINAD. To characterize the error, the theoretical  $SINAD_{REF}$  was calculated from the precisely known stimulating sine wave, and quantisation noise was achieved as a difference of the recorded samples and the precisely known stimulating sine wave by Eq. (10).  $SINAD_{LS}$  and  $SINAD_{ML}$  were calculated from sine wave recovered by LS and ML fit, respectively.

Fig. 10 shows the deviation of SINAD<sub>LS</sub> and SINAD<sub>ML</sub> from SINAD<sub>REF</sub>, respectively, as well as variance of repeated testing as a dependence on ADC noise  $\sigma$ . The results were achieved for nonlinear 8 bit ADC and the record length N=2048 samples taken by coherent sampling.



Fig. 10. Error of SINAD calculated from LS and ML fit for different variances of ADC noise.

The errors of SINAD calculated in the LS fit decrease with increasing noise. This can be explained by noticing that the input noise acts like dither, thus quantization noise is becoming more and more independent of the signal for larger noise amplitude.

Fig. 11. shows again results achieved for a nonlinear ADC and for different numbers of samples in a coherent record. Again, the error of the ML fit is strikingly smaller.



Fig. 11. Error of SINAD calculated from LS and ML fit for nonlinear ADC and different number of samples in coherent record.

Fig. 12. shows errors of SINAD calculated from ML and LS fits as functions of the maximum nonlinearity of the nonlinear ADC. Again, the ML method gives stable INL independent results within the whole tested range of ADC nonlinearity, while the error of LS increases with increasing nonlinearity. This can be explained by the fact that LS fitting

does not model the nonlinearity in any way, while ML fitting does, even if this performs approximate modelling only (2048 samples in the record, i.e. approximately 4 samples in a bin).



Fig. 12. Error of SINAD of LS and ML fit for different maximum ADC INLs.

To confirm that ML gives better estimation of input signal than LS fit and avoid possible errors caused by our models we decided to use also third party models. There has been developed and suggested a big variety of ADC models by various authors, e.g., behavioural models in [15], [16], [17], etc. For implementation in our software and experiments we chose the third-party ADC models for some fast ADCs [12] and simulation tool developed by Analog Devices as the leading vendor of ADCs on the world market with long-time experience and tradition in design and producing ADCs. Example of achieved results for AD9287 which INL is depicted in Fig. 13 is shown in Fig. 14. The results are similar to the results achieved in previous simulations, which utilised our ADC model. The differences between ML and LS fit, and the stimulus signal were smaller than those using our ADC model, due to the smaller INL of the model (Fig. 4 and Fig. 13). However the ML fit was always closer to the original stimulus (Fig. 14.).



Fig. 13. Nonlinear INL of an ADC model AD9287



Fig. 14. Comparison of LS and ML fits, with digitized and exact stimulus (zoomed time segment) for ADC model AD9287. The result of ML estimation is much closer to the true signal than that of the LS fit.

Finally, Fig. 15 shows average machine times required for the calculation of the ML fit for different numbers of samples in a record, for gradient-based method and for differential evolution. The fitted parameters were the parameters of sine wave and  $\sigma$ . Calculation was performed on a machine

- o for differential evolution: with processor Intel Core i5 (3,32GHz); 4G RAM, OS W7 64bit Professional, and developed fitting software was running under LabVIEW 2011 SP1 32bit environment.
- o for gradient-based method: with processor AMD V140 (2.30 GHz); 2GB RAM, OS W7 64bit Professional, developed fitting software was running under MATLAB 7.5.0.342 (R2007b)



Fig. 15. Average machine time used for the ML fit using the differential evolution algorithm in LabVIEW, and gradientbased method in Matlab. For comparison also the LS fitting time is depicted.

The DE is somewhat slower than the gradient based method. The fitting time depends mainly on number of samples.

#### 4. CONCLUSIONS

In this paper a comparative study of LS and ML fits for ADC testing was carried out. Two main aspects were investigated: (i) effectiveness and precision of sinewave fit from record if ADC input range is overloaded and (ii) influence of fitting method on SINAD testing. The achieved results show that while for linear ADC both methods give similar results for the studied conditions and applications, for nonlinear ADC the maximum likelihood fit clearly outperforms the LS fit. Besides, for overloading ADC the ML fit may be directly applied for whole record while using LS fit the record must be first pre-processed by excluding the overloading samples.

Using the numerical minimization method using the DE algorithm, the process of ML fit can be simply implemented in LabVIEW using built-in function GlobalOptimization.vi. For 1024 samples, the run time of the DE method was about 10 seconds on a standard PC, and a few seconds for the gradient-based method.

We have made the Matlab code available through the Internet [11]. In a short time also the LabVIEW code will be available online.

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