INTRODUCTION TO SIGN ERROR SPECTRAL OBSERVER

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I. Introduction

The sign error observer algorithm introduced in the paper is based on the relationship between the least mean square (LMS) [1] and resonator based observer algorithms [2][3], but utilizes the sign error LMS algorithm [4] for estimating the state variables of the observed system. Since the algorithm uses the signum of the error of the estimation, significant reduction in the amount of data required for the algorithm and in the computational demand can be achieved. Hence the utilization of this algorithm reduces design restrictions in systems with limited resources (e.g. bandwidth of communication channels).

Possible utilizations of the observer are the Fourier decomposition of signals and adaptive control. An application example is an active noise control (ANC) system [8] that uses wireless sensor network (WSN) for noise sensing [6]. This is a straightforward field for the deployment of this algorithm, since ANC systems require lots of sensors and relatively high sampling frequency taking into account the typical bandwidth of the WSN's radio standards (e.g. ZigBee) and the real time data transmission, so data reduction plays important role.

II. Review of the traditional resonator based observer structure

The resonator based observer was designed to follow the state variables of the so-called conceptual signal model [2]. The signal model is described as follows:

$$\mathbf{x}_{n+1} = \mathbf{x}_n\,; \qquad \mathbf{x}_n = [x_{i,n}]^{\mathrm{T}} \tag{1}$$

$$y_n = \mathbf{c}_n \cdot \mathbf{x}_n = \sum_{i=-L}^{L} c_{i,n} \cdot x_{i,n} \tag{2}$$

$$\mathbf{c}_n = [c_{i,n}];\; c_{i,n} = e^{j\cdot\omega_i\cdot n} = e^{j\cdot i\omega_1\cdot n},\; i=-L\ldots L\,, \tag{3}$$

where $\mathbf{x}_n$ is the state vector of the signal model at time step n, y_n is its output (the input of the observer), $\mathbf{c}_n$ represents the basis functions. To generate a real signal $\omega_{-i}=-\omega_i$ shall be satisfied. Obviously, in these cases the corresponding state variables shall form complex conjugate pairs. The conceptual signal model can be considered as a summed output of resonators which can generate any multisine with components up to the half of the sampling frequency. The corresponding observer is (see Fig. 1):

$$\hat{\mathbf{x}}_{n+1} = \hat{\mathbf{x}}_n + \mathbf{g}_n\,(y_n - \mathbf{c}_n \cdot \hat{\mathbf{x}}_n) = \hat{\mathbf{x}}_n + \mathbf{g}_n\,(y_n - y'_n) = \hat{\mathbf{x}}_n + \mathbf{g}_n\, e_{\mathrm{n}}; \qquad \mathbf{g}_n = [g_{i,n}]^{\mathrm{T}} = [r_i\, c^*_{i,n}]^{\mathrm{T}}, \tag{4}$$

where $\{\hat{\mathbf{x}}_n = [\hat{x}_{i,n}]^{\mathrm{T}};\, i=1\ldots N;\, N=2L+1\}$ is the estimated state vector, $\{r_k;\, k=1\ldots N\}$ are free parameters to set the poles of the system, and * denotes the complex conjugate. N is the number of harmonic components. Due to the complex exponentials, the channels of the observer can be considered as time-invariant systems with a pole on the unit circle. This is why they are called resonators. If the resonator poles are arranged uniformly on the unit circle, and $\{r_k={}^{1}/_{\mathrm{N}};\, k=1\ldots N\} \rightarrow \mathbf{g}_n = {}^{1}/_{N}\; \mathbf{c}_n^{\mathrm{H}}$ ($^{\mathrm{H}}$ denotes the conjugate transpose), the observer has finite impulse response, and the observer corresponds to the recursive discrete Fourier transform (RDFT) [2]. If the alignment of the resonators is not uniform, the settling is no longer deadbeat, but the system is still stable.

Since (4) corresponds to the formula of LMS, (4) can be interpreted as the state variables were updated by the complex LMS algorithm, where the reference signal is $\mathbf{c}_n$. Using this relationship between the observer and LMS [3] in the proposed new observer structure the sign error LMS (SE-LMS) algorithm is used for updating the state variable $\hat{\mathbf{x}}_n$.

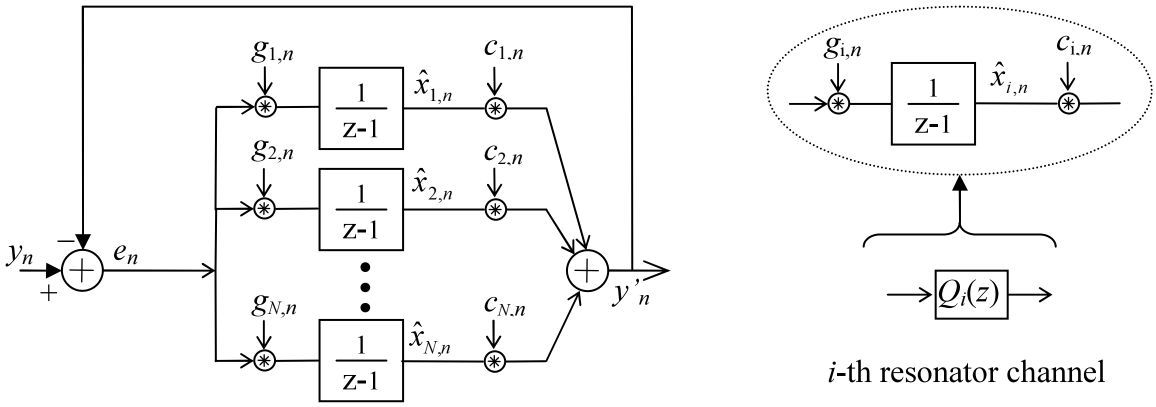

Figure 1: Basic configuration of the resonator based observer

III. The Sign Error Observer Structure

The proposed sign error structure can be seen in Fig. 2. The update procedure is the following:

$$\hat{\mathbf{x}}_{n+1} = \hat{\mathbf{x}}_n + \mathbf{g}_n \operatorname{sgn}(e_n); \qquad \mathbf{g}_n = [g_{k,n}]^{\mathrm{T}} = [\alpha c^*_{k,n}]^{\mathrm{T}} = \alpha \mathbf{c}_n^{\mathrm{H}}, \tag{5}$$

where $e_n = (y_n - y'_n)$ is the error of the estimation. $\operatorname{sgn}(x)=|x|/x$, i.e. $\operatorname{sgn}(x)=+1$ if $x>0$, -1 if $x<0$ and $\operatorname{sgn}(x)=0$ if $x=0$. It means that $v=1$ in Fig. 2 in the case of this simple sign error observer. This updating requires only the knowledge of the sign of the error, so it needs less computation, than the original algorithm—see (4)—, and the amount of data required for the operation is reduced. This is advantageous if it is implemented in systems with constrained resources. α is used for setting the transient and steady state behavior of the observer.

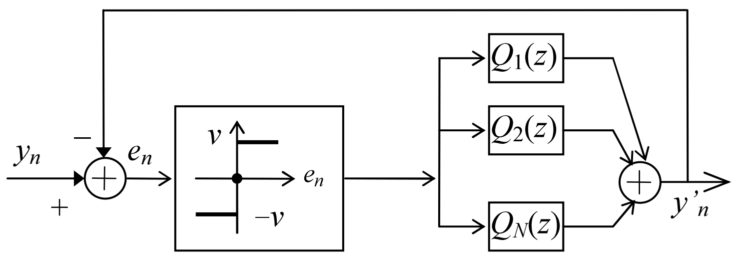

Figure 2: Basic configuration of the resonator based sign error observer

The steady state error of the observer can be determined by adapting the results in [4] for this structure:

$$E_{\mathrm{a}}(n) = \frac{1}{n}\sum_{k=0}^{n-1}|e_k| \;\le\; \frac{\|\mathbf{x}\|^2}{2\alpha n} + \frac{N}{2}\alpha, \tag{6}$$

where E_{a} is the absolute mean error. (6) implies that if $n\to\infty$ (system is in steady state), the average absolute error is bounded by $N\alpha/2$ that is proportional to the convergence parameter α.

The settling time M of the observer can be estimated by the recursive expansion of (5):

$$\hat{\mathbf{x}}_M = \sum_{j=0}^{M-1} \alpha \cdot \mathbf{c}_j^{\mathrm{H}} \cdot \operatorname{sgn}(e_j) + \hat{\mathbf{x}}_0 \,. \tag{7}$$

Taking the absolute value, and assuming that the initial state $\hat{\mathbf{x}}_0 = 0$ we get:

$$\|\hat{\mathbf{x}}_M\| = \left\|\sum_{j=0}^{M-1} \alpha \cdot \mathbf{c}_n^{\mathrm{H}} \cdot \mathrm{sgn}(e_j)\right\| \le \sum_{j=0}^{M-1} \left\|\alpha \cdot \mathbf{c}_n^{\mathrm{H}} \cdot \mathrm{sgn}(e_j)\right\| = \sum_{j=0}^{M-1} \alpha \cdot \left\|\mathbf{c}_n^{\mathrm{H}}\right\| = M\alpha\sqrt{N}. \tag{8}$$

From (7) with the assumption that $\hat{\mathbf{x}}_M \approx \mathbf{x}$ (the observer is in steady state at time instant M) the estimation of the settling time is:

$$M \ge \frac{\|\mathbf{x}\|}{\alpha\sqrt{N}}. \tag{9}$$

(6) and (9) pose contradictory conditions for the observer. The following section introduces the improved version of the observer which ensures fairly fast convergence with small steady state error.

IV. The Improved Sign Error Observer Structure

In order to resolve the above mentioned contradictory conditions an adaptive tuning of the convergence parameter is proposed:

$$\beta = \alpha v = \alpha \|\mathbf{e}_m\|_1 ; \quad \mathbf{e}_m = [e_m \; e_{m-1} \; \ldots \; e_{m-V+1}]^{\mathrm{T}}, \tag{10}$$

where β is the new convergence parameter. $v = \|\mathbf{e}_m\|_1$, $\mathbf{e}_m$ is a vector consisting of the last V values of the error signal at the time instant m when β is modified. $\|\cdot\|_1$ denotes the absolute value norm. It can be called normalized sign error spectral observer. The updating algorithm is the following:

$$\hat{\mathbf{x}}_{n+1} = \hat{\mathbf{x}}_n + \alpha \mathbf{c}_n^{\mathrm{H}} \cdot \|\mathbf{e}_m\|_1 \cdot \mathrm{sgn}(e_n) ; \quad \mathbf{g}_n = \alpha \|\mathbf{e}_m\|_1 \mathbf{c}_n^{\mathrm{H}}. \tag{11}$$

If the value of the error signal is high then v is also high, so the state variables are updated more radically (with larger steps), thus the convergence is faster. If the estimation error is low—the estimated and real value of $\mathbf{x}$ are near to each other—$\hat{\mathbf{x}}_n$ is updated with lower modifications so decreasing the error of the observation. These facts mean that the utilization of the norm of the error improves the behavior of the sign error observer. The frequently the parameter v is calculated the faster the convergence is. If $V = 1$, the original observer is obtained.

The optimal value of α in (10) and (11) can be calculated for the case when resonators are aligned uniformly and β is updated in each period of y_n. Let's denote the k-th period of the signal by k. These conditions mean that $V=N$, and $m=kN$ in (10), so $\mathbf{e}_m=\mathbf{e}_{kN}=[e_{kN} \ldots e_{kN-N+1}]$, since for uniformly aligned resonators the length of one period of the signal is N.

In these circumstances the observer algorithm minimizes $\|\mathbf{e}_m\|^2$, thus it makes the power (i.e. mean square) of one period of the error signal minimal if the optimal α is utilized:

$$\alpha_{\mathrm{opt}} = \frac{1}{N \cdot N_{\mathrm{NZ}}}, \tag{12}$$

where N_{NZ} is the number of nonzero elements of $\mathbf{e}_m$. In practice this result can be used as an initial value when the refreshing of the convergence factor is taken place with other period or resonators are placed unevenly.

For this structure the convergence of the algorithm depends on the properties of the signal. Let assume that α_{opt} is used. N step convergence can be achieved if all elements of the error signal $\mathbf{e}_{kN}$ in (10) have the same absolute value: $|e_i|=|e_j|$: $\forall i, j \in [kN \ldots kN - N + 1]$. In worst case the error signal is a periodic impulse: except of one dominant element of the period that is $e_j=A$, the other elements are nearly zero: $e_i \to 0$, but $|e_i|>0$ that is important in the calculation of $\mathrm{sgn}(e_i)$. In this case the ratio of the mean square values of consecutive error periods is: $\lambda = \left\|\mathbf{e}_{(k+1)N}\right\|^2 / \left\|\mathbf{e}_{kN}\right\|^2 = \left(1 - \frac{1}{N}\right)$. Using this worst case value of the decreasing ratio a higher bound for the settling time can be given. Let M denote the number of periods during which the power of error decreases to its ρ-th part. Using these conditions:

$$M \le -\frac{\lg(\rho)}{\lg(\lambda)}. \tag{13}$$

V. Results

The preliminary practical results with the introduced sign error spectral observer were achieved in a resonator based wireless active noise control (ANC) system [6][7]. ANC systems are special kind of control systems, where the plant to be controlled is an acoustic one [8]. The controller algorithm is a variant of the spectral observer [7], where the error signal e_n is the noise that is sensed by a microphone.

In our system the noise is sensed by a wireless sensor that samples the error signal (i.e. remaining noise), performs the calculation of the signum function and the norm of the error signal and sends the data to a DSP that implements the observer structure. The sampling frequency of the error signal is 1.8 kHz. Due to the utilization of the normalized sign error observer the amount of the data to be transmitted from the sensor to the DSP was one sixth than that in the case of normal observer. The reason is that instead of the current value of the signal only the sign of the error and the absolute norm of the error in V=32 samples long intervals were transmitted. The data reduction is important in this system because the bandwidth of the communication channel (250 kbps) is relatively low compared to the sampling frequency (some kilohertz). This kind of signal compression makes possible either the expansion of the number of sensors with the same sampling frequency, or the increase of the sampling frequency.

VI. Conclusions and future plans

This paper introduced a simple sign error and a normalized sign error spectral observer that can be deployed in systems with limited resources. These spectral observer algorithms are advantageous because they require reduced amount of information for the updating of the state variables, since the sign of the error signal can be represented by lower number of bits than the value of the error signal. The computational requirements can also be reduced since the multiplication with the sign of the error instead of the value of the error can be substituted by a simple addition or subtraction.

In the paper the transient and steady state properties of the algorithms were also derived. It was shown that the simple sign error observer requires tradeoff between the accuracy and speed of adaptation. In order to relieve these contradictory conditions a normalized sign error spectral observer was presented. This algorithm utilizes the first norm of the error signal for tuning the parameters of adaptation, and provides faster convergence with small steady state error. As a practical application a wireless active noise control system was introduced.

The aim of the future work is the extension of the structure on MIMO case, as well.

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