ENERGY-BASED SYNTHESIS OF TENSION MODULATION IN STRINGS

*Balázs Bank**

Dept. of Measurement and Information Systems,
Budapest University of Technology and Economics
Budapest, Hungary
bank@mit.bme.hu

ABSTRACT

Above a certain amplitude, the string vibration becomes nonlinear due to the variation of tension. An important special case is when the tension varies with time but spatially uniform along the string. The most important effect of this tension modulation is the exponential decay of the pitch (pitch glide). In the case of nonrigid string termination, the generation of double frequency terms and the excitation of missing modes also occurs, but this is perceptually less relevant for most of the cases. Several modeling strategies have been developed for tension modulated strings. However, their computational complexity is significantly higher compared to linear string models. This paper proposes efficient techniques for modeling the quasistatic part (short-time average) of the tension variation that gives rise to the most relevant pitch glide effect. The modeling is based on the linear relationship between the energy of the string and quasistatic tension variation. When this feature is added to linear string models, the computational complexity is increased by a negligible amount, leading to significant savings compared to earlier tension modulated string models.

1. INTRODUCTION

While the principal behavior of string instruments can be described by the linear wave equation, it cannot cover some important secondary effects. This is because above certain amplitude of vibration, the tension is not any more constant, leading to longitudinal string motion, which acts back to the transverse vibration. Because this nonlinearity comes from the geometry of the problem (the elasticity of the string material is assumed to be linear), it is called "geometric nonlinearity". Depending on the parameters of the string and the excitation force, the vibration can be classified into five different regimes [1]. This is depicted in Table 1, where the different classes are separated depending whether the transverse to longitudinal coupling (T→L), the longitudinal to transverse coupling (L→T), or the longitudinal inertial effects (L inertial eff.) are significant.

There are various available models [2, 3, 4], that are able to model the full effect of geometric nonlinearity, corresponding to "Bidirectional coupling" of Table 1. However, they are computationally demanding, and their accuracy is not always required, because for most of the musical instruments some effects of the nonlinear behavior can be neglected.

One important special case is "Tension modulation", when the transverse vibration acts on the longitudinal one and vice versa, but the inertial effects of the longitudinal vibration can be neglected.

	T → L	L → T	L inertial eff.
Linear motion			
Double freq. terms	×		
Tension modulation	×	×	
Longitudinal modes	×		×
Bidirectional coupling	×	×	×

Table 1: Main features of the different regimes of string behavior. The "×" sign means that the specific feature of vibration is significant.

This means that the longitudinal motion of the string immediately follows that of the transverse one to find equilibrium in the force along the string, and the longitudinal modes play no role. When this condition is met, the tension is spatially uniform along the string and can be directly computed from the transverse slope, as will be discussed in Sec. 2.

In practice, this is the case for loosely stretched strings where ratio of the longitudinal and transverse fundamental frequencies is large. Important examples are electric and steel-stringed acoustic guitars and ethnic instruments including the Finnish kantele and some oriental counterparts. The most important perceptual effect of tension modulation is the pitch glide, meaning that the pitch of the string decreases as the sound decays. The pitch glide coming from tension modulation has an even more significant effect in drums. In the classical drum set, tom-toms have a characteristic pitch glide, but many other percussion instruments produce this effect. This paper concentrates on string modeling, but the basic idea is also applicable to drum synthesis.

Many different models have been presented for tension modulated strings (see Sec. 3), but their computational complexity is significantly higher compared to efficient linear string models. In this paper, a new tension modulation methodology is proposed where the quasistatic part (short-time average) tension is approximated from the energy of the string. The method leads to significant computational savings and makes it possible to include tension modulation in string modeling also in less powerful computational environments (such as mobile phones, games, etc.).

The paper is organized as follows: first the theory describing tension modulation is summarized in Sec. 2, followed by the prior work on related sound synthesis methods in Sec. 3. Section 4 relates the energy of the string to the tension, while Sec. 5 provides two efficient modeling techniques. Motivated by its importance in string modeling, a third alternative is proposed for the digital

* Part of this work was supported by the EEA and Norway Grants and the Zoltán Magyary Higher Education Foundation.

waveguide in Sec. 6. Section 7 concludes the paper and gives future research directions.

2. TENSION MODULATION

In the most general case, the wave equation for a single transverse polarization of the lossless and nondispersive string is

$$\mu \frac{\partial^2 y}{\partial t^2} = \frac{\partial \left(T(x,t) \frac{\partial y}{\partial x} \right)}{\partial x}, \tag{1}$$

where $y = y(x,t)$ is the string displacement, $T(x,t)$ is the tension of the string, and μ is the linear mass density. Note that in the case of linear string vibrations, the tension is constant $T(x,t) = T_0$.

When the inertial effects of longitudinal modes can be neglected, the tension is spatially uniform along the string $T(x,t) = \overline{T}(t)$ and can be directly computed from the elongation of the string according to the Hooke's law [5]:

$$\overline{T}(t) = T_0 + QS[(L'(t) - L)/L], \tag{2}$$

where $L'(t)$ is the actual length of the string and L is the minimum length at equilibrium, S is the cross-section area of the string, and Q is the Young's modulus. The overline in $\overline{T}$ emphasizes that the tension is spatially uniform along the string. The length L' equals the length of the curve $y(x,t)$ for a given t and $0 \leq x \leq L$ and is given by

$$L' = \int_0^L \sqrt{1 + \left(\frac{\partial y}{\partial x} \right)^2} dx \approx L + \frac{1}{2} \int_0^L \left(\frac{\partial y}{\partial x} \right)^2 dx. \tag{3}$$

The substitution of Eq. (3) into Eq. (2) gives

$$\overline{T} = T_0 + \frac{1}{2} \frac{QS}{L} \int_0^L \left(\frac{\partial y}{\partial x} \right)^2 dx. \tag{4}$$

Inserting Eq. (4) into Eq. (1) yields

$$\mu \frac{\partial^2 y}{\partial t^2} = \left[T_0 + \frac{1}{2} \frac{QS}{L} \int_0^L \left(\frac{\partial y}{\partial x} \right)^2 dx \right] \frac{\partial^2 y}{\partial x^2} \tag{5}$$

which is the Kirchhoff-Carrier equation used by most of the papers as a starting point (although sometimes extended to the z polarization). It can be seen that Eq. (5) has the same form as the linear wave equation but now the constant tension T_0 is replaced by a time dependent term (parametric nonlinearity).

Legge and Fletcher [6] have investigated the intermodal coupling due to the tension modulation. That is, how a specific transverse mode can gain energy from another transverse mode. First, the transverse displacement is written in its modal form

$$y(x,t) = \sum_{n=1}^{\infty} y_n(t) \sin\left(\frac{n\pi x}{L} \right), \tag{6}$$

where $y_n(t)$ is the instantaneous amplitude of the transverse mode n. Then, Eq. (6) is inserted into Eq. (4), giving

$$\overline{T}(t) = T_0 + \frac{\pi^2 QS}{4L^2} \sum_{n=1}^{\infty} n^2 y_n^2(t). \tag{7}$$

After the excitation, the string modes decay exponentially, thus the instantaneous amplitudes $y_n(t)$ become exponentially decaying sinusoidal functions

$$y_n(t) = A_n \sin(\omega_n t + \varphi_n) e^{-\frac{t}{\tau_n}}, \tag{8}$$

which yields the following expression for tension:

$$\overline{T}(t) = T_0 + \frac{\pi^2 QS}{8L^2} \sum_{n=1}^{\infty} n^2 A_n^2 [1 - \cos(2\omega_n t + 2\varphi_n)] e^{-\frac{2t}{\tau_n}}. \tag{9}$$

The first time-dependent part of Eq. (9) is a quasistatic increase of tension

$$T_{\text{qs}} = \frac{\pi^2 QS}{8L^2} \sum_{n=1}^{\infty} n^2 A_n^2 e^{-\frac{2t}{\tau_n}}, \tag{10}$$

which decays slowly. This leads to a proportional increase in the wave speed c and the modal frequencies, giving a relative change of $\sqrt{(T_0 + T_{\text{qs}})/T_0}$. This shift decreases as a function of time, leading to a pitch glide which is usually the most relevant perceptual effect of tension modulation.

The second part contains the double frequency terms

$$T_{\text{df}} = -\frac{\pi^2 QS}{8L^2} \sum_{n=1}^{\infty} n^2 A_n^2 \cos(2\omega_n t + 2\varphi_n) e^{-\frac{2t}{\tau_n}}, \tag{11}$$

leading to a continuous modulation of tension, built up of sinusoidal functions having double the frequencies of transverse modes. The amplitude of this modulation decays exponentially, and the decay times of its components are the half compared to that of the originating transverse modes.

Substituting Eq. (9) into Eq. (1) and concentrating on the effects of double frequency terms leads us to the observation that the different transverse modes cannot efficiently exchange energy if the string is rigidly terminated [6]. As a practical result, for rigid terminations, the double frequency terms do not have an effect, and only the quasistatic part Eq. (10) is relevant.

If the bridge is not infinitely rigid, but has the admittance $Y(\omega)$ at $x = L$, then the spatial distribution of the transverse modes are not anymore orthogonal. In this case the force acting on the bridge contains the frequencies $2\omega_n \pm \omega_m$ and all the modes can gain energy from the bridge motion. Strong coupling arises when the excitation and resonance frequencies are near, thus, the mode p can gain energy from modes $p = 2m \pm n$, as $\omega_p \approx 2\omega_m \pm \omega_n$ [6]. For a more realistic bridge, when the string passes the bridge at an angle, the tension Eq. (9) directly appears in the bridge movement. This means that the double frequency terms $2\omega_n$ can directly excite any of the transverse modes. Naturally, effective excitation will arise when $p = 2n$, as in this case the excitation frequency $2\omega_n$ will be close to the resonance frequency ω_p of mode p [6].

We have to note that the periodic tension variation Eq. (11) can lead to very significant audible effects for some musical instruments. This is because the force on the bridge in the longitudinal string direction equals the tension. Therefore, tension variation can excite the body of the instrument by periodically pulling and releasing the bridge, leading to double frequency terms. The relevance of this effect depends on how effectively the longitudinal bridge force is radiated by the instrument body. While for most of the western instruments this radiation is not significant, it contributes to the characteristic sound of some special instruments. This is the case for the kantele (traditional Finnish instrument) where the string force in the longitudinal direction is effectively coupled to the body, because the strings are terminated by a vertical tuning peg, instead of a proper bridge [7, 8].

As an example, a simulated electric guitar string vibration is displayed in Fig. 1. The example is computed by a finite difference string model, and the excitation is modeled by applying a triangle-shaped initial displacement distribution with a peak value of 5 mm. The quasistatic tension T_{qs} (dashed line in Fig. 1(b)) is calculated by a running average filter with an averaging length equaling the time period of the tension variation. The tension at rest T_0 is 100 N, and the maximal tension variation is around 7.5 N, leading to a pitch change of 3.6 %, which is around a half semitone and can be heard easily. The slow rise of the quasistatic tension (dashed line) between 0 and 5 ms is the side effect of the running average computation, since it has zero input at negative times.

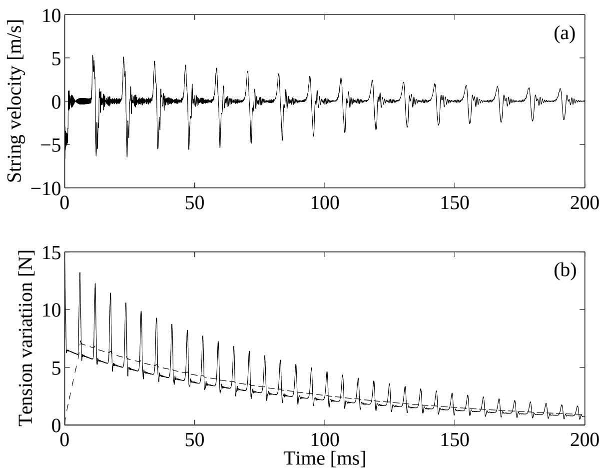

Figure 1: *Simulated low E ($f_0 = 82.4$ Hz) electric guitar string excited by a 5 mm pluck. String velocity at the pickup position (a) and tension variation $T - T_0$ (b). The quasistatic part of the tension variation T_{qs} is displayed by dashed line in (b).*

3. PRIOR WORK IN TENSION MODULATED STRING SYNTHESIS

We have seen that the tension can be decomposed to a quasistatic and a periodic part. The first attempt to synthesize the effect of tension modulation has concentrated on the periodic variation [7]. While usually the most relevant perceptual effect of tension modulation is the pitch glide coming from the quasistatic part, for some instruments, like the kantele cited above, the periodic tension variation is highly significant. For modeling, the output of the transverse string model (implemented by a digital waveguide) is lead to a second-order nonlinearity and a lowpass filter, and the result is mixed with the string output [7]. The nonlinearity adds the required double-frequency components, but some unwanted sum- and difference frequencies too. In [7] the main motivation was to add a reinforcement to the second harmonic. As a result, efficient lowpass filtering could be used after the nonlinearity, suppressing the unwanted peaks.

For modeling the complete (both periodic and quasistatic) temporal modulation of tension, various methods have been presented for the different string modeling paradigms. The most efficient linear string modeling technique is the digital waveguide [9, 10]. In this case the effect of tension variation can be taken into account by varying the delay line length, which is done by a variable allpass filter at the termination [11, 8]. The first step is the computation of string tension at each time instant by approximating Eq. (4), from which the instantaneous propagation speed is obtained. Then, the required length change of the total delay line is computed by the boxcar integration of the instantaneous speed change. As the length of the integration is the pitch period, this is practically the same as calculating the propagation speed from the quasistatic tension variation $T_{\mathrm{qs}}(t)$. The computationally most demanding part of the algorithm is the calculation of the tension Eq. (4), increasing the load significantly compared to linear string models, even when Eq. (4) is approximated as a sparse sum. Recently, an energy conserving variation of the technique have been presented in [12]. A computationally even more demanding, but more accurate method is distributing the variable length delays between the delay elements [13, 14, 15].

For finite-difference modeling, it is relatively straightforward to implement the tension modulation by changing the tension parameter of a linear string model according to Eq. (4). An energy-conserving variation was presented in [16], which is beneficial as the stability of the model is guaranteed. A modal-based tension modulation string model have been presented by [17], where the model parameters are derived by the Functional Transformation Method. An energy conserving modal method was proposed in [18] .

4. RELATION TO ENERGY

For rigid string terminations, it is only the quasistatic variation of tension that has an effect. For example, the terminations of the electric guitar and the electric bass can be considered perfectly rigid. Even for nonrigid string terminations, where the periodic tension variation can play some role in theory, the most prominent effect is the pitch glide due to the quasistatic tension variation (with the exception of some exotic instruments like the kantele). Therefore, it is reasonable to concentrate on the modeling of the quasistatic part.

In this section, we derive the relationship between the string energy and the quasistatic part of the tension. As will be shown later, the energy of the string can be estimated at lower computational complexity, thus, the quasistatic tension can be computed with less operations than by computing the elongation of the string.

4.1. Basic equations

The energy of the string has two parts: one is the kinetic energy due to the movement of the string, and the other is the potential energy due to the stretching of the string. The kinetic energy of a string segment dx at position x is computed as

$$E_{\mathrm{k}}(x,t) = \frac{1}{2}\mu dx\; v(x,t)^2, \tag{12}$$

where μdx is the mass of the element and $v(x,t) = \partial y(x,t)/\partial t$ is the velocity of the string motion. The total kinetic energy is the integration of Eq. (12) over the string length L:

$$E_{\mathrm{k}}(t) = \frac{1}{2}\int_0^L \mu \left(\frac{\partial y(x,t)}{\partial t}\right)^2 dx. \tag{13}$$

The potential energy can be computed from the elongation of the infinitesimally small string elements. The initial length of the element is dx, and it is stretched to ds during string motion. Because the tension is spatially uniform along the string, all the elements are stretched by equal amount, that is, ds is not space dependent. Therefore, we may consider the whole string in the longitudinal direction as a single elastic (spring-like) element. Thus, the potential energy stored in the "string spring" becomes

$$E_{\mathrm{p}}(t) = \frac{1}{2}K(L' - L_0)^2, \tag{14}$$

where L_0 is the length of the unstretched string (without any initial tension T_0), and K is the spring constant QS/L_0. Note that L is referring to the initial length of the string with tension T_0, and L' is the actual length during vibration. Thus, the change from L_0 to L' has two steps: first, the length changes by $L - L_0$ when the string is stretched by the force T_0 (the string is put on the instrument and tuned), and then by $\Delta L = L' - L$ during vibration. Thus, the potential energy becomes

$$E_{\mathrm{p}}(t) = \frac{1}{2}K(L - L_0)^2 + K(L - L_0)\Delta L + \frac{1}{2}\Delta L^2. \tag{15}$$

In Eq. (15) the first term is the potential energy of the the stretched, but not vibrating string (string at equilibrium) $E_{\mathrm{p},0} = 1/2K(L - L_0)^2$, and the initial tension is $T_0 = K(L - L_0)$. By omitting the last (second-order) term the total potential energy can be approximated by

$$E_{\mathrm{p}}(t) \approx E_{\mathrm{p},0} + T_0\Delta L = E_{\mathrm{p},0} + T_0\frac{1}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx. \tag{16}$$

Since the initial potential energy $E_{\mathrm{p},0}$ is just an additive constant, it will be neglected in the rest of the paper. Accordingly, $E_{\mathrm{p}}(t)$ will refer to the additional potential energy of the string due to its vibration. Thus, the sum of the kinetic and potential energy $E = E_{\mathrm{k}} + E_{\mathrm{p}}$ is the total energy of the string that comes from string vibration, and the string at rest leads to $E = 0$.

4.2. Modal formulation

If the transverse displacement is expressed in the modal form, and Eq. (6) is inserted into Eq. (13), the kinetic energy becomes

$$E_{\mathrm{k}}(t) = \frac{1}{2}\mu L\sum_{n=1}^{\infty} v_n(t)^2 = \frac{1}{2}\mu L\sum_{n=1}^{\infty}\left(\frac{\partial y_n(t)}{\partial t}\right)^2, \tag{17}$$

where $v_n(t) = \partial y_n(t)/\partial t$ is the instantaneous velocity of mode n. Writing the instantaneous amplitudes in the form of exponentially decaying sinusoids as in Eq. (8) gives

$$E_{\mathrm{k}}(t) = \frac{1}{4}\mu L\sum_{n=1}^{\infty}\omega_n^2 A_n^2(1 + \cos(2\omega_n t + 2\varphi_n))e^{-\frac{2t}{\tau_n}}. \tag{18}$$

By using Eq. (6), for the potential energy $E_{\mathrm{p}}(t)$, we obtain

$$E_{\mathrm{p}}(t) = \frac{1}{4}\frac{T_0\pi^2}{L}\sum_{n=1}^{\infty} n^2[y_n(t)]^2. \tag{19}$$

Again writing the instantaneous amplitudes in the form of exponentially decaying sinusoids gives

$$E_{\mathrm{p}}(t) = \frac{1}{8}\frac{T_0\pi^2}{L}\sum_{n=1}^{\infty} n^2 A_n^2(1 - \cos(2\omega_n t + 2\varphi_n))e^{-\frac{2t}{\tau_n}}, \tag{20}$$

which is very similar to Eq. (18). The only difference is the sign of the cosine terms, and apparently the leading constants. However, it turns out that the leading constants actually equal. The modal frequencies ω_n of a nondispersive string are computed as

$$\omega_k = k\frac{\pi}{L}\sqrt{\frac{T_0}{\mu}}. \tag{21}$$

When Eq. (21) is inserted into Eq. (18) and added to Eq. (20), the cosine terms cancel out, giving

$$E(t) = E_{\mathrm{k}}(t) + E_{\mathrm{p}}(t) = \frac{1}{4}\frac{T_0\pi^2}{L}\sum_{n=1}^{\infty} n^2 A_n^2 e^{-\frac{2t}{\tau_n}}, \tag{22}$$

By comparing Eq. (22) with the quasistatic part of the tension Eq. (10), it turns out that $T_{\mathrm{qs}}(t)$ is a scaled version of the total energy

$$T_{\mathrm{qs}}(t) = \frac{QS}{2LT_0}E(t), \tag{23}$$

which is the fundamental outcome of this section.

4.3. Finite difference example

The energy of the same electric guitar string as in Fig. 1 is displayed in Fig. 2(a), showing a similar decay compared to the quasistatic tension. The small wrinkles on the curve are due to the approximation in energy computation in Eq. (16).

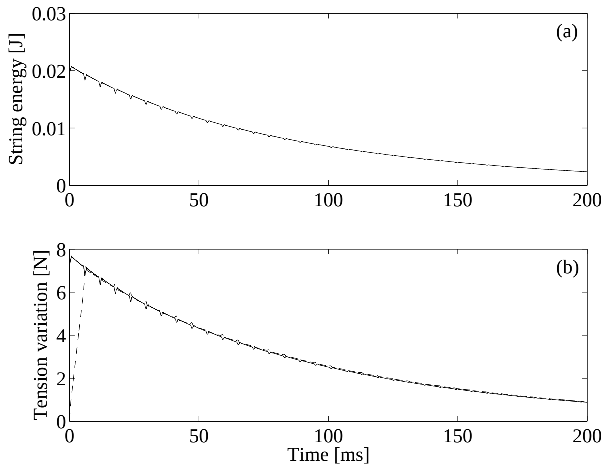

Figure 2: *Simulated low E ($f_0 = 82.4$ Hz) electric guitar string excited by a 5 mm pluck, (a): string energy, (b): quasistatic tension. In (b), the dashed line displays the quasistatic tension computed by a running average filter, and the solid line displays the quasistatic tension computed from the energy of the string by Eq. (23).*

As can be seen in Fig. 2(b), the quasistatic tension computed from the energy of the string by Eq. (23) (solid line) is very close to the short time average of the tension variation (dashed line). Note that the difference in the range of 0–5 ms is due to the error introduced by the running average (see also in Fig. 1). Therefore, the energy-based tension variation (solid line) shows the correct behavior.

5. EFFICIENT MODELING

We have seen in Sec. 4 and in Fig. 2 that the quasistatic tension can be accurately computed from the string energy. In the previous string models mentioned in Sec. 3, tension computation is a separate model block which varies the parameters of a linear string model. Therefore, the earlier models can be easily modified by substituting the tension computation block with an energy computation block and a simple scaling. This will not yet lead to computational savings because computing the string energy in a straightforward way takes a similar number of operations as computing the tension. However, the computational complexity of the energy computation can be decreased significantly, as will be shown in the following sections.

5.1. Downsampled energy computation

As can be seen in Fig. 2, the energy and the quasistatic tension are slowly changing signals, in contrast to the total tension variation (see Fig. 1(b) solid line). Therefore, it is sufficient to compute the string energy at a lower rate (say, at every 1–10 ms), and the continuous energy curve is obtained by linearly interpolating between the computed points. As a result, the average load of energy computation becomes negligible compared to the time-variant linear string model itself.

5.2. Energy storage model

By looking at Fig. 2, one can notice that the quasistatic tension decays exponentially. This is also clear from Eq. (10) if we assume that the decay times of the modes τ_n are the same for all n. In reality, the decay times of the modes are different, but usually for the lowest, dominant modes they are in the same order, and the resulting energy- and quasistatic tension curves are close to exponential. Therefore, it seems tempting to model the energy storage of the string as a first order lowpass filter, whose decay time is determined by the dissipation speed of the string. If the string excitation is modeled by an initial displacement and/or velocity distribution, such as the triangle-shaped initial displacement for plucking, then the initial value of the lowpass filter is set to the initial energy of the string.

In a more general case of a dynamic excitation (like a physical model of a pluck or strike), the input of the lowpass filter is the amount of energy $\Delta E[n]$ that has entered to the string, so the string energy $E[n]$ is computed as:

$$E[n] = \Delta E[n] + gE[n-1], \qquad (24)$$

where g is a constant determining the rate of energy decay. The energy input $\Delta E(n)$ can be computed from the energy change of the excitation $\Delta E_{\mathrm{exc}}[n]$, since from the law of conservation of energy we have

$$\Delta E[n] = -\Delta E_{\mathrm{exc}}[n]. \qquad (25)$$

Discussing realistic models of plucking is out of the scope of the paper. Therefore, the method is illustrated by a hammer strike, since striking is one of the simplest form of excitation. (Note that the piano hammer example is only for illustrative purposes. In real pianos, the rise of longitudinal components is the dominant effect, and the pitch glide is negligible. Therefore, for piano modeling, other models should be used, see, e.g., [19]).

The piano hammer is usually modeled by a small mass m_{h} connected to a nonlinear spring with an exponent P_{h} and stiffness K_{h} [20, 21]:

$$F_{\mathrm{h}}[n] = F(\Delta y) = \begin{cases} K_{\mathrm{h}}(\Delta y)^{P_{\mathrm{h}}} & \text{if} \Delta y > 0 \\ 0 & \text{if} \Delta y \leq 0 \end{cases}, \qquad (26)$$

where $F_{\mathrm{h}}[n]$ is the interaction force and $\Delta y[n]$ is the compression of the hammer felt. The energy of the hammer is computed as

$$E_{\mathrm{exc}}[n] = \frac{1}{2} m_{\mathrm{h}} v_{\mathrm{h}}^2 + \int_0^{\Delta y} F_{\mathrm{h}} dx = \\ = \frac{1}{2} m_{\mathrm{h}} v_{\mathrm{h}}^2 + K \frac{\Delta y^{p+1}}{p+1}, \qquad (27)$$

where v_{h} is the velocity of the hammer. Since Eq. (27) is simple to compute, it can be calculated for every time instant. For more complex excitation models, we may compute $E_{\mathrm{exc}}[n]$ at a lower rate (e.g., every tenth sample), but this is usually not necessary because most of the excitation models are zero dimensional, leading to simple calculations. Naturally, the energy of the excitation block has to be computed only during the excitation.

Then, the energy input to the string is computed as the energy change of the excitation block:

$$\Delta E[n] = -(E_{\mathrm{exc}}[n] - E_{\mathrm{exc}}[n-1]). \qquad (28)$$

In the case of a lossy excitation mechanism, such as a hysteretic hammer, the dissipation of the damping element has to be subtracted from the energy input $\Delta E[n]$.

Figure 3 shows the energy decay of a string excited by a 15 m/s hammer strike. The dashed line displays the string energy both in linear (a) and logarithmic scale (b). The energy shows a linear decay on a logarithmic scale, as expected. The dashed line is the estimated energy computed by the energy storage model of Eq. (24), and the energy of the hammer was computed by Eq. (27). The g parameter of Eq. (24) was estimated by fitting a line on the logarithmic energy plot, but it can also be estimated from the decay time of the string. The small discrepancy between the curves is most probably due to the fact that the losses of the string are described correctly by Eq. (24) only during the free decay, and not during the excitation. However, the estimated and real energy curves are close enough for the accuracy required by sound synthesis.

Note that this physically informed tension computation technique can also be used with other, not physics-based synthesis paradigms, such as sampling and spectral models. This is because we have a separate energy storage model for the string, instead of computing the energy from the string model itself. If the energy of the excitation signal is precomputed as a function of its parameters (e.g., hammer velocity), the energy can be computed by Eq. (24), from which the tension, and finally the pitch can be obtained. Then, this pitch parameter can drive any synthesis model, e.g., based on spectral synthesis or simple sample playback.

6. MODELING WITH DIGITAL WAVEGUIDES

The digital waveguide technique [9, 10] has a great importance in string modeling, since it is the most efficient way of simulating one dimensional wave propagation. As a result, most of the work in tension modulated strings has concentrated on developing digital waveguide models. While the two methods proposed in Sec. 5 are

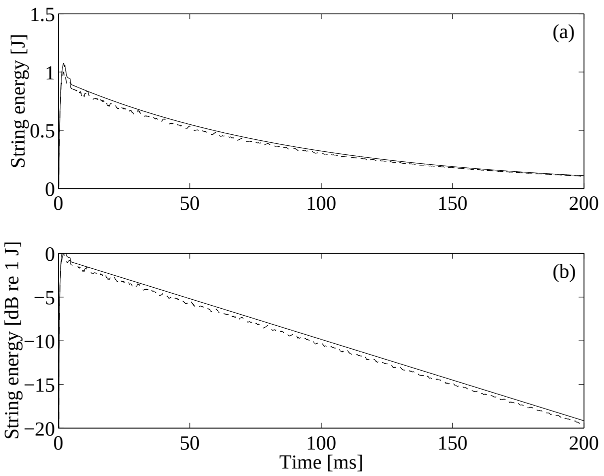

Figure 3: *Simulation of a string excited by a piano hammer: sting energy in linear (a) and logarithmic (b) scale. The dashed line is the string energy, and the solid line is the estimated string energy computed by the energy storage model Eq. (24).*

easily applicable in the case of digital waveguides as well, here a third alternative is proposed that computes the string energy $E[n]$ more precisely than the physically informed technique of Sec. 5.2 at a similar computational cost.

6.1. Energy of the digital waveguide

Digital waveguide modeling is based on the traveling wave solution of the ideal (lossless and nondispersive) wave equation:

$$y(x,t) = y^+(x-ct) + y^-(x+ct), \quad (29)$$

where y^+ and y^- can be considered as two traveling waves, which retain their shape during their movement. The function y^+ is the wave going to the right, and the function y^- is the wave going to the left direction.

The velocity of the string is obtained as

$$v(x,t) = \frac{\partial y(x,t)}{\partial t} = v^+(x-ct) + v^-(x+ct) = \\ -c\frac{\partial y^+(x-ct)}{\partial t} + c\frac{\partial y^-(x+ct)}{\partial t}, \quad (30)$$

where v^+ and v^- are the velocity waves traveling to the right and left, respectively. Once the string displacement and velocity are known, the string energy $E(t)$ can be computed by the help of Eqs. (13) and (16). This results in the simple expression

$$E(t) = \mu \int_0^L [v^+(x-ct)]^2 dx + \mu \int_0^L [v^-(x+ct)]^2 dx, \quad (31)$$

which is basically the integration of the squared velocities of the two traveling waves. Note that the usual $1/2$ leading term is missing because Eq. (31) computes not only the kinetic energy, but the total energy of the string, and for a single traveling wave, the total energy is the double of the kinetic energy (the potential and kinetic energies equal).

In digital waveguides, the traveling waves are stored in delay lines, whose content is shifted to model wave propagation. For simplifying the energy computation, it is beneficial to choose the velocity as the wave variable for the digital waveguide, instead of string displacement. (Note that in most of the cases velocity is the chosen wave variable anyway, because it is related to the force by the string impedance.) In this case, the energy is simply computed as the squared sum of the delay elements

$$E[n] = \mu \Delta x \sum_0^M (v^+[m,n])^2 + (v^-[m,n])^2, \quad (32)$$

where $v^+[m,n]$ is the content of the upper delay line at spatial position m and time instant n, and $v^-[m,n]$ is the content of the lower delay line in a dual delay line implementation. The constant $\Delta x = c\Delta t = L/M$ is the spatial step size of the digital waveguide, where $\Delta t = 1/f_s$ is the sampling time and M is the length of the string in samples (the total number of delay elements is $2M$). If the digital waveguide is implemented as a single delay-loop model [22], the computation remains the same but now the squared sum is performed for one delay line only.

6.2. Efficient energy computation

In addition to the ideas proposed in Sec. 5, we may take advantage of some properties of digital waveguides for efficient energy computation. We have seen that the energy of the digital waveguide can be computed by a simple squared sum of the delay elements. In the case of an ideal string, the content of the delay line simply circulates in the delay-line buffer, and the energy remains constant (there is no energy loss in the delay line). For modeling the lossy string, the losses are usually consolidated to one point in the loop, as displayed in Fig. 4, where $H_r(z^{-1})$ is the loss filter. The only point where losses can occur is at the loss filter, and at every time step it is only one value in the delay line that changes (the one which passes the loss filter). All the other values are simply shifted. Therefore, we can easily compute the loss occurring, $\Delta E[n]$, by computing the energies of the samples entering and leaving the loss filter, and taking their difference, as displayed in Fig. 4. Then, the string energy $E[n]$ is computed as the integration of $\Delta E[n]$. From the energy, we compute the quasistatic tension by a simple scaling according to Eq. (23), from which the propagation speed $c[n]$ is obtained as

$$c[n] = c_0 \sqrt{\frac{T_0 + T_{qs}[n]}{T_0}} \approx c_0 \left(1 + \frac{1}{2}\frac{T_{qs}[n]}{T_0}\right), \quad (33)$$

where $c_0 = \sqrt{T_0/\mu}$ is the nominal propagation speed. The actual length of the delay lines is computed as

$$D = \frac{L}{c[n]} f_s \approx D_0 \left(1 - \frac{1}{2}\frac{T_{qs}[n]}{T_0}\right) \quad (34)$$

where $D_0 = L f_s / c_0$ is the nominal length of the delay line. The integer part of $D[n]$ is implemented by the delay line z^{-M} in Fig. 4, while the fractional part is realized by the fractional delay filter FD, which is typically a first order allpass filter or a Lagrange FIR filter [23].

It is important to note that in [11, 8] the length of the delay line is computed by the boxcar integration of the instantaneous propagation speed. This is not necessary here, since $c[n]$ is already the averaged version of the instantaneous propagation speed, because it is computed from the quasistatic tension $T_{qs}[n]$ instead of the total tension $T[n]$. This explains why the present method

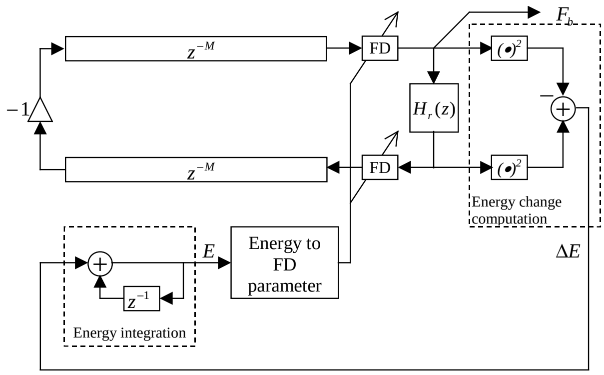

Figure 4: *Energy-based tension modulation modeling with a digital waveguide.*

can be more efficient than that of [11, 8]: instead of computing the total variation of tension (which takes a significant amount of computation) and averaging it by boxcar integration, it computes the average tension variation directly.

Figure 4 does not contain any excitation block for clarity. If the excitation is modeled as an initial velocity distribution of the string (e.g., a stepwise function for plucking), then the initial value of the energy $E[0]$ is set to the energy computed from the initial shape, and this is corrected by $\Delta E[n]$ in each time instant. In the case of a separate excitation block, such as a hammer model connected to the string, the initial string velocity and the initial energy are zero. The string is set to motion by the excitation force, which is added to the two delay lines with an appropriate scaling at a certain spatial position m. The energy change of the string due to the excitation force is computed in the same way as for the loss filter: the energy before the summation point is compared to the energy after the summation point.

In general, at each point where the lossless wave propagation in the digital waveguide is broken by a junction (being a filter, an additional input, or a more complex excitation block), the energy before and after the junction has to be computed, and this difference is the energy change for that junction. The total energy change $\Delta E[n]$ is the sum of the energy changes at the junctions:

$$\Delta E[n] = \mu \Delta x \sum_{i=1}^{I} (J_{\mathrm{out},i}[n])^2 - (J_{\mathrm{in},i}[n])^2, \qquad (35)$$

where $J_{\mathrm{in},i}[n]$ is the input and $J_{\mathrm{out},i}[n]$ is the output of junction i, and I is the number of junctions. Naturally, the dimension of $J_{\mathrm{in},i}$ and $J_{\mathrm{out},i}$ is velocity (m/s). In a typical scenario of a single excitation point, $I = 3$ since the excitation enters the whole delay line at two points (one point for the upper and one point for the lower line), in addition to the loss filter. However, after the excitation period when the string decays freely, only the energy change at the "loss junction" has to be computed.

Compared to the physically informed energy storage model of Sec. 5.2, the present method provides slightly more precise results because it does not assume exponential energy decay, and the g parameter of Eq. (24) does not have to be estimated. The largest advantage is that there is no need to estimate the energy of the excitation block, since the energy input coming from the excitation is directly computed by Eq. (35). The computational complexity of the two methods is in the same order.

7. CONCLUSION AND FUTURE WORK

Tension modulation in strings has two effects: a continuos decrease of pitch (pitch glide) and the nonlinear coupling of transverse modes. For rigidly terminated strings (such as the electric guitar), mode coupling does not occur, and even for nonrigid string terminations in most western instruments, the pitch glide is the most important phenomenon. Coming from this observation, this paper has presented an efficient modeling methodology based on the linear relationship between the energy of the string and the quasistatic part (short-time average) of the tension variation. Basically, the computationally heavy tension calculation block in earlier string models is substituted with a more efficient energy computation block and a simple scaling. As a result, the model precisely synthesizes the pitch glide occurring in tension modulated strings, while the additional computational complexity compared to linear string models is negligible, in contrast to earlier tension modulated string models.

Future research includes the precise comparison with earlier methods in terms of sound quality and computational complexity. If needed, the effect of nonlinear mode coupling could also be added by leading the string output to a second-order nonlinearity, as was done for kantele synthesis in [7].

It seems feasible to model the tension modulation in membranes based on the energy of the membrane. Thus, the efficient modeling of pitch glides in drum synthesis is an important field of future research. Since membrane modeling takes significantly more computational time than string modeling, the computational savings provided by the energy-based tension computation can be even more important than for the string.

The interested reader may listen to the sound examples at http://www.mit.bme.hu/∼bank/publist/dafx09.

8. ACKNOWLEDGEMENTS

The author is thankful for the comments of Federico Avanzini, Jyri Pakarinen, Stefano Zambon, and the anonymous reviewers, which have helped to improve the paper significantly.

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